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The spinorial geometry of supersymmetric heterotic string backgrounds

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Abstract: We determine the geometry of supersymmetric heterotic string backgrounds for which all parallel spinors with respect to the connection $\hat{\nabla}$ with torsion H, the NS$\otimes$NS three-form field strength, are Killing. We find that there are two classes of such backgrounds, the null and the timelike. The Killing spinors of the null backgrounds have stability subgroups $K \ltimes \mathbb{R}^8$ in $Spin(9,1)$, for $K = Spin(7)$, $SU(4)$, $Sp(2)$, $SU(2) \times SU(2)$ and $\{1\}$, and the Killing spinors of the timelike backgrounds have stability subgroups G_2, $SU(3)$, $SU(2)$ and $\{1\}$. The former admit a single null $\hat{\nabla}$-parallel vector field while the latter admit a timelike and two, three, five and nine spacelike $\hat{\nabla}$-parallel vector fields, respectively. The spacetime of the null backgrounds is a Lorentzian two-parameter family of Riemannian manifolds B with skew-symmetric torsion. If the rotation of the null vector field vanishes, the holonomy of the connection with torsion of B is contained in K. The spacetime of time-like backgrounds is a principal bundle P with fibre a Lorentzian Lie group and base space a suitable Riemannian manifold with skew-symmetric torsion. The principal bundle is equipped with a connection λ which determines the non-horizontal part of the spacetime metric and of H. The curvature of λ takes values in an appropriate Lie algebra constructed from that of K. In addition dH has only horizontal components and contains the Pontrjagin class of P. We have computed in all cases the Killing spinor bilinears, expressed the fluxes in terms of the geometry and determine the field equations that are implied by the Killing spinor equations.

Keywords: Superstrings and Heterotic Strings, Supergravity Models, Superstring Vacua, Flux compactifications.

Contents

1. Introduction

It has been known for some time that the geometry of supersymmetric heterotic string backgrounds resembles that of Riemannian manifolds that appear in the Berger classification list and admit parallel spinors. This is because the gravitino Killing spinor equation is a parallel transport equation for a metric connection $\hat{\nabla}$ with torsion given by the NS⊗NS three-form field strength H. Therefore the solutions of the gravitino Killing spinor equation are characterized by the holonomy of $\hat{\nabla}$, $\mathrm{hol}(\hat{\nabla})$. This holonomy group is contained in the stability subgroup G of the parallel spinors in a suitable spin group. Berger classified the irreducible Riemannian manifolds using the holonomy of the Levi-Civita connection. Similarly, the holonomy of the Levi-Civita connection ∇ of these Riemannian manifolds which in addition admit parallel spinors is again contained in the stability subgroup of the spinors. Because of this, it has been expected that there must be a relation between the holonomies of $\hat{\nabla}$ that appear in supersymmetric heterotic string backgrounds and those of the Levi-Civita connection ∇ of Berger irreducible Riemannian manifolds that admit parallel spinors as both are contained in the stability subgroups of the parallel spinors. It turns out that there is such a relation but there are also differences because the spacetime of supersymmetric heterotic backgrounds is a Lorentzian and not a Riemannian manifold. So the stability subgroups of the parallel spinors are in $Spin(n-1,1)$ instead of $Spin(n)$ which is suitable for Riemannian manifolds. In addition, the heterotic string supergravity has two more Killing spinor equations associated with the dilatino and gaugino supersymmetry transformations.

The geometry of manifolds that admit a metric connection with skew-symmetric torsion has been extensively investigated in the literature. Such geometries appear in the context of supersymmetric one- and two-dimensional sigma models, see e.g. [1 − 4]. They have also been explored as supersymmetric solutions of the common sector of type II theories and heterotic supergravity, and their properties have been examined using the Killing spinor bilinear forms, see e.g. [5 − 7]. Deformations of these geometries due to higher curvature corrections of the heterotic string have been investigated in e.g. [5, 8 − 11]. It has been recognized some time ago that these geometries with torsion are closely related to the standard geometries, like Kähler, Calabi-Yau and hyper-Kähler, see e.g. [12 − 18], and they have found applications in the geometry of black-hole moduli spaces [19 − 21]. More recently, these geometries with torsion have been studied using the Gray-Hervella classification techniques [22], see e.g. [23 − 27]. So far in the applications of these geometries in the context of ten-dimensional supergravity, it has been assumed that the spacetime is a product, $\mathbb{R}^{9-n,1} \times X_n$, and the non-trivial part of the geometry is that of the Riemannian manifold X_n. We shall not make such an assumption and we shall find that the spacetime geometry of supersymmetric heterotic backgrounds is not always a product.

In this paper, we shall use the method developed in [28] to systematically investigate all possible geometries of supersymmetric heterotic string backgrounds. The parallel transport equation , $\hat{\nabla}\epsilon = 0$, implies that

$$\hat{R}\epsilon = 0 , \tag{1.1}$$

where $\hat{R}$ is the curvature of $\hat{\nabla}$ and takes values in $\mathfrak{spin}(9,1)$. If the Killing spinors ϵ have a non-trivial stability subgroup G in $Spin(9,1)$, $G \subset Spin(9,1)$, then the holonomy of $\hat{\nabla}$ must be a subgroup of G, $\mathrm{hol}(\hat{\nabla}) \subseteq G$. The Killing spinors are the singlets of G in the decomposition of the Majorana-Weyl representation $\Delta^+_{\mathbf{16}}$ of $Spin(9,1)$ under G. On the other hand if the stability subgroup is $\{1\}$, then the holonomy of $\hat{\nabla}$ is the identity and

$$\hat{R} = 0 . \tag{1.2}$$

Therefore either the Killing spinors are singlets of a proper subgroup of $G \subset Spin(9,1)$ or $\hat{R} = 0$. In the former case, we shall give all the spinors which are singlets of a subgroup of $Spin(9,1)$. Some of these are related to the parallel spinors that exist on the manifolds that appear in the Berger classification list and have been presented in [29]. However in our case the stability subgroups are somewhat different because the spacetime is a Lorentzian manifold. In the latter case, the spacetime is parallelizable with respect to the $\hat{\nabla}$ connection. Using this, we shall show that the spacetime is a Lorentzian metric Lie group and that $\hat{\nabla}$ is a parallelizable connection.

The investigation of the gaugino Killing spinor equations $F\epsilon = 0$ is similar to that of the curvature condition $\hat{R}\epsilon = 0$. This is because the Clifford element F lies in the $\mathfrak{spin}(9,1)$ subspace of the Clifford algebra. If the spinors ϵ that satisfy $F\epsilon = 0$ have a non-trivial stability subgroup G in $Spin(9,1)$, then the curvature F takes values in the Lie algebra $\mathrm{g} \subseteq spin(9,1)$ of G. If the stability subgroup is $\{1\}$, then $F = 0$ and the gauge connection is flat. In addition, the expression $F\epsilon$ for any spinor ϵ can be read off from that for the gravitino Killing spinor equation, in particular from the part that contains the spin connection. Because of this, we shall not explore further the supersymmetry conditions that arise from the gaugino Killing spinor equation.

The dilatino Killing spinor equation is somewhat different from the gravitino and gaugino Killing spinor equations. In particular, there is no understanding of the solutions of the dilatino Killing spinor equation in terms of Lie subalgebras of $\mathfrak{spin}(9,1)$ similar to the one presented above for the gravitino and gaugino Killing spinor equations. However it can be analyzed using representation theory. It is also known that there are backgrounds[1] with spinors which solve the gravitino but not the dilatino Killing spinor equation. *Because of this, we shall restrict our attention to those backgrounds for which all the solutions of the gravitino Killing spinor equation are also solutions of the dilatino one*, i.e. all $\hat{\nabla}$-parallel spinors are Killing. In the terminology of [31], these are the maximally supersymmetric G-backgrounds, where G is the stability subgroup of the Killing spinors.

It is convenient to characterize the supersymmetric heterotic string backgrounds in terms of the number of supersymmetries they admit, which we denote with N, and the stability subgroup of the Killing spinors G in $Spin(9,1)$, [32, 33]. We shall show that the stability subgroups G of the Killing spinors are either compact groups K, $G = K$, for $K = G_2(N = 2), SU(3)(N = 4), SU(2)(N = 8), \{1\}(N = 16)$ or $G = K \ltimes \mathbb{R}^8$, for $K = Spin(7)(N = 1), SU(4)(N = 2), Sp(2)(N = 3), SU(2) \times SU(2)(N = 4), \{1\}(N = 8)$, where N denotes the number of supersymmetries. In the former case the stability subgroups G are those expected from the Berger classification list. The latter case has no Riemannian analogue and is due to the Lorentzian signature of spacetime but the subgroups K appear in the Berger classification list. The Killing spinors are chiral with respect to a suitable chirality projector of a Clifford algebra $\mathrm{Cliff}(\mathbb{R}^8) \subset \mathrm{Cliff}(\mathbb{R}^{9,1})$.

We shall show that the supersymmetric backgrounds for which the Killing spinors have a compact stability subgroup admit a time-like and at least two space-like parallel vector spinor bilinears[2] with respect to $\hat{\nabla}$. Because of this, we shall refer to them as time-like backgrounds. The commutator of the parallel vector fields does not necessarily vanish and the structure constants depend on the NS$\otimes$NS three-form field strength H. If one imposes the condition that the algebra $\mathfrak{h}$ spanned by parallel vectors constructed from the spinor bilinears closes under Lie brackets, then the spacetime M for $K \neq \{1\}$ is (locally) a principal bundle $M = P(\mathcal{H}, B, \pi)$ equipped with the connection λ, where $\mathcal{H}$ is a Lie group with Lie algebra $\mathfrak{h}$ and base space B which is the space of orbits of the parallel vector fields. The backgrounds with $K = \{1\}$ are maximally supersymmetric and it has been shown in [34] that the spacetime is locally isometric to $\mathbb{R}^{9,1}$, $H = 0$ and $\Phi = \text{const}$. The spacetime metric and torsion can be written as

$$ds^2 = \eta_{ab}\, \lambda^a \lambda^b + g^h \ ,$$

[1]Examples of such manifolds are Lorentzian metric groups for which all spinors are parallel with respect to the left-invariant connection but such backgrounds typically preserve 1/2 of the supersymmetry because half of these spinors do not solve the dilatino Killing spinor equation.

[2]Since the Killing spinors are parallel with respect to $\hat{\nabla}$, all the Killing spinor form bilinears α are also parallel with respect to $\hat{\nabla}$, $\hat{\nabla}\alpha = 0$.

$$H = \frac{1}{3}\eta_{ab}\,\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\,\lambda^a \wedge \mathcal{F}^b + H^h \ , \tag{1.3}$$

where g^h and H^h are the horizontal components of the metric and H, $\mathcal{F}$ is the curvature of the connection λ and η is a Lorentzian invariant metric on $\mathcal{H}$. The dilaton Φ depends only on the coordinates of B. In addition

$$dH = \eta_{ab}\,\mathcal{F}^a \wedge \mathcal{F}^b + dH^h \,. \tag{1.4}$$

Therefore dH contains a representative of the first Pontrjagin class of P. The Killing spinor equations impose restrictions on $\mathcal{H}$, $\mathcal{F}$ and the geometry of the base space B. The gravitino Killing spinor equation implies that the spacetime admits a K-structure compatible with a metric connection with skew-symmetric torsion, $\mathrm{hol}(\hat{\nabla}) \subseteq K$. There are three kinds of conditions that arise from the dilatino Killing spinor equation. One set of conditions imposes restrictions on the Lie group $\mathcal{H}$, another set of conditions suitably restricts the curvature $\mathcal{F}$ of the connection λ, and the third set of conditions implies restrictions on the geometry of B.

In particular, for $K = G_2$, $\mathcal{H}$ has Lie algebra either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$; for $K = SU(3)$, $\mathcal{H}$ is a four-dimensional Lorentzian Lie group but otherwise unrestricted; for $K = SU(2)$, $\mathcal{H}$ is a six-dimensional Lorentzian metric Lie group but the dilatino Killing spinor equation imposes restrictions of its structure constants which we determine.

The second set of conditions of the dilatino Killing spinor equation implies that the connection λ is a $\mathfrak{k}$ instanton, i.e. $\mathcal{F}$ takes values in the Lie algebra $\mathfrak{k}$ of K. This is the case for all K apart from $K = SU(3)$ and $\mathcal{H}$ non-abelian, where $\mathcal{F}$ satisfies the Donaldson conditions and takes values in $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$.

The base space B has dimension, $\dim B = 7, 6, 4$ for $K = G_2, SU(3), SU(2)$, respectively. In addition B admits a *conformally balanced and integrable K-structure, and a compatible metric connection $\hat{\tilde{\nabla}}$ with skew-symmetric torsion $\tilde{H}$*, $H^h = \pi^* \tilde{H}$, i.e. $\mathrm{hol}\,(\hat{\tilde{\nabla}}) \subseteq K$. This is the case for all K apart from $K = SU(3)$ and $\mathcal{H}$ non-abelian, where B admits an $SU_c(3) = SU(3) \times_Z R$-structure, where $R = U(1)$ or $R = \mathbb{R}$ and Z is a discrete group. The additional R twist is due to a one-dimensional representation ρ of $\mathcal{H}$ and the associated line bundle $L = P \times_\rho \mathbb{C}$. The conformally balanced structure is due to the fact that a Lee form of the K-structure of B is related to the exterior derivative of the dilaton as consequence of the conditions that arise from the dilatino Killing spinor equation. An integrability of the K-structure is also implied by the dilatino Killing spinor equations. This is suitably defined for all K. For example, if $K = SU(3)$, then the associated almost complex structure is integrable and B is a complex manifold. Furthermore, if $K = G_2$ and $\mathcal{H}$ is abelian, then dilatino Killing spinor equation also requires that $d\tilde{\varphi}$ is orthogonal to $\star\tilde{\varphi}$, where $\tilde{\varphi}$ is a G_2 invariant form on B. In the non-abelian case, this is not the case and the inner product $(d\tilde{\varphi}, \star\tilde{\varphi})$ is related to the structure constants of $\mathcal{H}$. In all the above cases, the NS$\otimes$ NS three-form H is determined by the form Killing spinor bilinears and the metric of the spacetime. In addition, the integrability conditions of the Killing spinor equations imply all the field equations provided that the Bianchi identities of H and F are satisfied.

Similarly, we shall show that the backgrounds for which the Killing spinors have stability subgroup $K \ltimes \mathbb{R}^8$, for $K = Spin(7)(N=1), SU(4)(N=2), Sp(2)(N=3), SU(2) \times SU(2)(N=4), \{1\}(N=8)$, admit a single null parallel one-form spinor bilinear κ with respect to $\hat{\nabla}$. Because of this, we shall refer to them as null backgrounds. If one adapts coordinates with respect to the null Killing vector field X associated to κ, $X = \partial/\partial u$, then the metric and torsion can be written as

$$\begin{aligned} ds^2 &= 2\mathrm{e}^-\mathrm{e}^+ + \delta_{ij} e^i_I e^j_J dy^I dy^J \ , \\ H &= \mathrm{e}^+ \wedge d\mathrm{e}^- + \frac{1}{2}(H^{\mathfrak{k}}_{-ij} + H^{\mathfrak{k}^\perp}_{-ij})\, \mathrm{e}^- \wedge e^i \wedge e^j + \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k \ , \end{aligned} \tag{1.5}$$

where $\kappa = \mathrm{e}^- = (dv + m_I dy^I)$ and $\mathrm{e}^+ = du + V dv + n_I dy^I$, and $H^{\mathfrak{k}}_{-ij}$ and $H^{\mathfrak{k}^\perp}_{-ij}$denote the components of H_{-ij} in the subalgebra $\mathfrak{k}$ of K and its orthogonal complement in $\Lambda^2(\mathbb{R}^8)$, $\Lambda^2(\mathbb{R}^8) = \mathfrak{k} \oplus \mathfrak{k}^\perp$. Moreover, $H^{\mathfrak{k}^\perp}_{-ij} = 2\,\Omega^{\mathfrak{k}^\perp}_{-,ij}$ in a suitably chosen frame. The Killing spinor equations determine all components of the NS⊗NS flux H in terms of the form Killing spinor bilinears and the spacetime metric apart from the component $H^{\mathfrak{k}}_{-ij}$. In addition, they imply that $d\kappa = d\mathrm{e}^-$ takes values in $\mathfrak{k} \oplus_s \mathbb{R}^8$, where $\oplus_s$ denotes semi-direct sum of Lie algebras. A consequence of this is that the null parallel vector field leaves invariant the $K \ltimes \mathbb{R}^8$-structure of spacetime.

In all null cases, the spacetime admits a codimension eight integrable foliation with leaves a manifold B. For generic backgrounds, B admits a K-structure which is not compatible with the induced metric connection $\hat{\tilde{\nabla}}$ with torsion. However, if $d\kappa = 0$, then B is a conformally balanced integrable manifold with a K-structure and compatible connection $\hat{\tilde{\nabla}}$ with torsion, i.e. $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq K$. The conformally balanced and integrability properties are consequences of the dilatino Killing spinor equations and are defined in a way similar to those of the space B in the timelike backgrounds we have mentioned above. We have also shown that the Killing spinor equations imply all field equations apart from the E_{--} component of the Einstein equations, the LH_{-A} components of the two-form gauge potential and the LF_- component of the field equations of the gauge connection provided that all the Bianchi identities are satisfied.

We also apply our results to investigate some properties of the Killing spinor equations of the common sector of type II supergravities. We find that the IIA and IIB common sectors should be treated separately because despite many similarities there are also differences. We mostly focus on the IIB common sector and investigate the supersymmetry conditions of backgrounds with two supersymmetries. We show that there are five distinct cases to examine described by the stability subgroups of the Killing spinors.

This paper is organized as follows: in section 2, we state the field and Killing spinor equations of heterotic supergravity and describe the integrability conditions of the latter. In section 3, we find the stability subgroups G of spinors in $Spin(9,1)$ and give the G-invariant spinors (singlets) in the Majorana-Weyl representation $\Delta^+_{\mathbf{16}}$ of $Spin(9,1)$. In section 4, we describe the parallel spinors and forms of supersymmetric backgrounds. We argue that there is always a basis up to a local Lorentz transformation such that the parallel spinors are constant. In section 5, we determine the geometry of $N=1$ backgrounds. In section 6, we give the geometry of $N=2$ $SU(4) \ltimes \mathbb{R}^8$-backgrounds. In section 7, we describe the geometry of $N = 2$ G_2-backgrounds. In section 8, we investigate the geometry of $N = 3$ backgrounds. In section 9, we determine the geometry of $N = 4$ $SU(3)$- and $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$-backgrounds. In section 10, we describe the geometry of $N = 8$ $SU(2)$- and $\mathbb{R}^8$-backgrounds. In section 11, we show that $\hat{\nabla}$-parallelizable backgrounds are Lorentzian metric Lie groups. In section 12, we apply our results to examine the supersymmetric solutions of the common sector of type II supergravities. In section 13, we give our conclusions. In appendix A, we describe the spinors in terms of forms and compute the form spinor bilinears for all singlets of a subgroup $G \subset Spin(9,1)$ in $\Delta^+_{\mathbf{16}}$. In appendix B, we present the linear systems associated with the Killing spinor equations of the heterotic supergravity.

2. Fields and spinors

2.1 Field and Killing spinor equations

The bosonic fields of heterotic supergravity are the metric g, the NS⊗NS three-form field strength H, the dilaton scalar Φ, and the gauge connection A with curvature F. The field and Killing spinor equations of the heterotic string receive string α' corrections which can be computed either from a sigma model beta function or from string amplitude calculations. The field equations in the string frame to lowest order in α' are

$$\begin{aligned} E_{MN} = R_{MN} - \frac{1}{4} H_{PQM} H^{PQ}{}_N + 2\nabla_M \partial_N \Phi &= 0 \, , \\ LH_{PQ} = \nabla_M (e^{-2\Phi} H^M{}_{PQ}) &= 0 \, , \\ L\Phi = \nabla^2 \Phi - 2g^{MN} \partial_M \Phi \partial_N \Phi + \frac{1}{12} H_{MNR} H^{MNR} &= 0 \, , \\ LF_N = \hat{\nabla}^M (e^{-2\Phi} F_{MN}) &= 0 \, , \end{aligned} \tag{2.1}$$

where ∇ is the Levi-Civita connection of the metric g. The field equation for the dilaton is implied from those of the metric and two-form gauge potential B associated with H, $H = dB$, up to a constant. The Killing spinor equations are

$$\begin{aligned} \hat{\nabla} \epsilon &= 0 \, , \\ (\Gamma^M \partial_M \Phi - \frac{1}{12} \Gamma^{MNP} H_{MNP}) \epsilon &= 0 \, , \\ F_{MN} \Gamma^{MN} \epsilon &= 0 \, , \end{aligned} \tag{2.2}$$

where $\hat{\nabla} = \nabla + \frac{1}{2} H$, $\nabla_M \epsilon = \partial_M \epsilon + \frac{1}{4} \Omega_{M,AB} \Gamma^{AB} \epsilon$,

$$\hat{\nabla}_N Y^M = \nabla_N Y^M + \frac{1}{2} H^M{}_{NR} Y^R \, , \tag{2.3}$$

and ϵ is a Majorana-Weyl spinor of positive chirality, i.e. ϵ is described by forms of even degree. In what follows, we shall denote the spin connection of the $\hat{\nabla}$ covariant derivative with $\hat{\Omega}$.

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2.2 Integrability conditions

It is well-known that some of the field equations of supersymmetric backgrounds can be implied by the Killing spinor equations. To find which field equations are implied, one has to investigate the integrability conditions of the Killing spinor equations. In the case of heterotic supergravity, these integrability conditions are, see also [35],

$$
\begin{aligned}
[\hat\nabla_M, \hat\nabla_N]\epsilon &= \frac{1}{4}\hat R_{MN,AB}\Gamma^{AB}\epsilon = 0 \ , \\
[\hat\nabla_M, F_{RS}\Gamma^{RS}]\epsilon &= 0 \ , \\
[\hat\nabla_M, \partial_N\Phi\Gamma^N - \frac{1}{12}H_{NPQ}\Gamma^{NPQ}]\epsilon &= 0 \ , \\
[F_{RS}\Gamma^{RS}, \partial_N\Phi\Gamma^N - \frac{1}{12}H_{NPQ}\Gamma^{NPQ}]\epsilon &= 0 \, .
\end{aligned}
\tag{2.4}
$$

Multiplying the first expression above with Γ^N, using appropriately the remaining integrability conditions and the identity

$$
\begin{aligned}
g^{MN}\partial_M\Phi\partial_N\Phi\epsilon - \frac{1}{24}H_{MNR}H^{MNR}\epsilon - \frac{1}{2}\partial_M\Phi H^M{}_{ST}\Gamma^{ST}\epsilon \\
+ \frac{1}{16}H^S{}_{MN}H_{SPQ}\Gamma^{MNPQ}\epsilon = 0 \ ,
\end{aligned}
\tag{2.5}
$$

one finds that

$$
\begin{aligned}
\hat R_{MA,BC}\Gamma^A\Gamma^{BC}\epsilon = -2E_{MN}\Gamma^N\epsilon - e^{2\Phi}LH_{MN}\Gamma^N\epsilon - \frac{1}{6}BH_{MABC}\Gamma^{ABC}\epsilon &= 0 \ , \\
L\Phi\epsilon - \frac{1}{4}e^{2\Phi}LH_{MN}\Gamma^{MN}\epsilon - \frac{1}{48}BH_{MNPQ}\Gamma^{MNPQ}\epsilon &= 0 \ , \\
\frac{1}{3}BF_{MNP}\Gamma^{MNP}\epsilon + 2e^{2\Phi}LF_N\Gamma^N\epsilon &= 0 \ ,
\end{aligned}
\tag{2.6}
$$

where $BH_{MNPQ} = 4(dH)_{MNPQ}$ and $BF_{MNR} = 3\nabla_{[M}F_{NR]}$. To the order of α' that we have stated the field equations above, the Bianchi identity of H implies that $BH = dH = 0$. However, if the heterotic string anomaly is included and so schematically, $BH \sim \alpha'(\mathrm{tr}R^2 - \mathrm{tr}F^2) + \mathcal{O}(\alpha'^2)$, then, for consistency, one has to include the two-loop correction to the field equations [9, 10].

A remarkable property of (2.6) is that if one imposes the Bianchi identities of H and F, $BH = 0$ and $BF = 0$, respectively, then the remaining equations are up to quadratic order in gamma matrices. As a result, it is straightforward to construct the linear systems associated with the integrability conditions from that of the Killing spinor equations. These linear systems are similar to those investigated in the context of M-theory and IIB supergravity in [31].

3. Stability subgroup of spinors in $Spin(9,1)$

As we have mentioned in the introduction, the Killing spinors of supersymmetric backgrounds with $\hat R \neq 0$ are singlets of the holonomy group $\mathrm{hol}(\hat\nabla)$ of $\hat\nabla$. In addition, the holonomy group in every case is a subgroup of the stability subgroup of the Killing spinors in $Spin(9,1)$. Therefore, we have to determine all the spinor singlets of the subgroups[3] of $Spin(9,1)$. This analysis closely resembles that of determining the parallel spinors of manifolds with special holonomy which has been presented in [29]. However, there are some differences that arise because the spacetime is a Lorentzian manifold.

3.1 One spinor

There is one type of orbit of $Spin(9,1)$ with stability subgroup $Spin(7) \ltimes \mathbb{R}^8$ in Δ^+_{16}. The proof of this has been given in [38] but we shall repeat the steps here because they are useful for determining the stability subgroups of more than one spinor. Consider the spinor

$$1 + e_{1234} \,. \tag{3.1}$$

The stability subgroup of this spinor in $Spin(9,1)$ is $Spin(7) \ltimes \mathbb{R}^8$ as it can be seen by solving the infinitesimal invariance equation

$$\lambda_{AB}\Gamma^{AB}(1 + e_{1234}) = 0 \ , \tag{3.2}$$

where λ parameterizes the spinor transformations. This computation is most easily done in the pseudo-Hermitian basis that we have given in (A.9). It is easy to see that the above condition implies that the parameters are restricted as

$$\lambda_{\bar\alpha\bar\beta} = \frac{1}{2}\epsilon_{\bar\alpha\bar\beta}{}^{\gamma\delta}\lambda_{\gamma\delta} \ , \qquad \lambda_{\alpha\bar\beta}g^{\alpha\bar\beta} = \lambda_{-+} = \lambda_{+\alpha} = \lambda_{+\bar\alpha} = 0 \ , \tag{3.3}$$

where $\epsilon_{\bar1\bar2\bar3\bar4} = 1$. Observe that the parameters $\lambda_{-\alpha}$ and $\lambda_{-\bar\alpha}$ are complex conjugate to each other but otherwise unconstrained. The group that leaves invariant $1 + e_{1234}$ has Lie algebra $\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$ and so find that the stability subgroup is $Spin(7) \ltimes \mathbb{R}^8$.

Having established this, we decompose Δ^+_{16} under the stability subgroup $Spin(7)$ as

$$\Delta^+_{16} = \mathbb{R} < 1 + e_{1234} > + \Lambda^1(\mathbb{R}^7) + \Delta_8 \ , \tag{3.4}$$

where the singlet $\mathbb{R}$ is generated by $1+e_{1234}$, $\Lambda^1(\mathbb{R}^7)$ is the vector representation of $Spin(7)$ which is spanned by the spinors associated with two-forms in the directions $e_1, \dots, e_4$ and $i(1 - e_{1234})$, and Δ_8 is the spin representation of $Spin(7)$ which is spanned by the rest of spinors which are of the type $\Gamma^+\eta$, η is a spinor generated by the odd forms in the directions $e_1, \dots, e_4$. Therefore the most general spinor in Δ^+_{16} can be written as

$$\eta = a(1 + e_{1234}) + \theta_1 + \theta_2 \ , \tag{3.5}$$

where $\theta_1 \in \Lambda^1(\mathbb{R}^7)$ and $\theta_2 \in \Delta_8$. First we assume that $a \neq 0$. In this case, there are two cases to consider depending on whether θ_2 vanishes or not. If $\theta_2 = 0$, since $Spin(7)$ acts with the vector representation on $\Lambda^1(\mathbb{R}^7)$, it is always possible to choose $\theta_1 = ib(1 - e_{1234})$. The most general spinor in this case then is

$$\eta = a(1 + e_{1234}) + ib(1 - e_{1234}) \,. \tag{3.6}$$

[3] We only consider connected subgroups of $Spin(9,1)$ as stability subgroups for spinors and our computations are restricted on the Lie algebra level. However spinors can admit disconnected stability subgroups and these are applicable to non-simply connected manifolds [36, 37].

However, it is easy to see that this spinor is in the same orbit as $1+e_{1234}$, e.g. observe that

$$\eta = h e^{\psi\Gamma_{16}}(1+e_{1234}) \;, \tag{3.7}$$

where $h^2 = a^2+b^2$ and $\tan\psi = b/a$. Next suppose that θ_2 does not vanish. If $\theta_2 \neq 0$, there is always a $Spin(7)$ transformation such that $\theta_2 = c\Gamma^+(e_1+e_{234})$. This is because $Spin(7)$ acts transitively on the S^7 in Δ_8 and the stability subgroup is G_2, $Spin(7)/G_2 = S^7$. In addition G_2 acts transitively on the S^6 in $\Lambda^1(\mathbb{R}^7)$ with stability subgroup $SU(3)$. So it can always be arranged such that $\theta_1 = ib(1-e_{1234})$. Therefore the most general spinor in this case is

$$\eta = a(1+e_{1234}) + ib(1-e_{1234}) + c\Gamma^+(e_1+e_{234})\,. \tag{3.8}$$

However observe that this spinor is in the same orbit of $Spin(9,1)$ as $1+e_{1234}$. Indeed

$$\eta = e^{\frac{b}{2c}\Gamma^-\Gamma^6} e^{\frac{c}{a}\Gamma^+\Gamma^1} a(1+e_{1234})\,. \tag{3.9}$$

So, we find that if $a\neq 0$, then there is one orbit represented by $a(1+e_{1234})$. It remains to investigate the case where $a=0$. In this case, it is straightforward to see that the orbit can always be represented by $c\Gamma^+(e_1+e_{234})$. In turn this spinor is in the same orbit of $Spin(9,1)$ as $\frac{c}{\sqrt{2}}(1+e_{1234})$ as it can seen by acting on the latter with the element $\Gamma_5\Gamma_1$ of $Spin(9,1)$. As a consequence, the stability subgroup of $c\Gamma^+(e_1+e_{234})$ is again $Spin(7)\ltimes\mathbb{R}^8$. Therefore, there is only one type of orbit of $Spin(9,1)$ in Δ^+_{16} which can be represented with $a(1+e_{1234})$. To conclude, the Killing spinor of backgrounds with one supersymmetry can be chosen, up to a Lorentz rotation for the fluxes, such that

$$\epsilon = f(1+e_{1234}) \;, \tag{3.10}$$

where f a spacetime function.

3.2 Two spinors

There are two types of $N=2$ backgrounds distinguished by the stability subgroup of the Killing spinors. To see this, we choose the first spinor to be $\epsilon_1 = a_1(1+e_{1234})$ with stability subgroup $Spin(7)\ltimes\mathbb{R}^8$. Then we decompose Δ^+_{16} as in (3.4).

One option is to take the second Killing spinor $\epsilon_2 \in \Lambda^1_7$. It turns out that $Spin(7)$ acts transitively on the sphere in $\Lambda^1_7 = \Lambda^1(\mathbb{R}^7)$ and so we can take $\epsilon_2 = a_2 i(1-e_{1234})$. The stability subgroup in $Spin(9,1)$ of both ϵ_1 and ϵ_2 is $SU(4)\ltimes\mathbb{R}^8$. Moreover $\Delta_8 = \Lambda^1_4(\mathbb{C}^4)\oplus\Lambda^3_4(\mathbb{C}^4)$ under $SU(4)$ and so there are no additional singlets. Therefore one class of $N=2$ backgrounds are those for which the Killing spinors are

$$\begin{aligned} \epsilon_1 &= f(1+e_{1234}) \;, \\ \epsilon_2 &= g_1(1+e_{1234}) + ig_2(1-e_{1234}) \;, \end{aligned} \tag{3.11}$$

with stability subgroup $SU(4)\ltimes\mathbb{R}^8$.

Next suppose that $\epsilon_2 \in \Delta_8$. $Spin(7)$ acts transitively on the sphere S^7 in the spinor representation Δ_8 with stability subgroup G_2. Because of this, the second Killing spinor can be chosen as $\epsilon_2 = b_2\Gamma^+(e_1+e_{234})$. In addition $\Lambda^1_7(\mathbb{R}^7)$ is an irreducible representation of G_2

and so there are no additional singlets. Therefore the second Killing spinor can be chosen as $\epsilon_2 = a_2(1+e_{1234}) + b_2\Gamma^+(e_1+e_{234})$. However in this case, it can be simplified further using the additional $\mathbb{R}^8$ invariance of $1+e_{1234}$. In particular observe that $e^{\frac{a_2}{b_2}\Gamma^-\Gamma^1} b_2\Gamma^+(e_1+e_{234}) = a_2(1+e_{1234}) + b_2\Gamma^+(e_1+e_{234})$. Therefore, we can take as a second spinor $\epsilon_2 = a_2\Gamma^+(e_1+e_{234})$. To summarize, another class of $N=2$ backgrounds are those for which the Killing spinors are

$$\begin{aligned} \epsilon_1 &= f(1+e_{1234})\ , \\ \epsilon_2 &= g\Gamma^+(e_1+e_{234})\ , \end{aligned} \tag{3.12}$$

which have stability subgroup G_2.

It remains to take the second spinor to be an element of $\Lambda^1_7(\mathbb{R}^7) \oplus \Delta_8$. One can again use the $Spin(7)$ invariance of ϵ_1 to set the component of the second spinor ϵ_2 in Δ_8 to be along the direction $\Gamma^+(e_1+e_{234})$. As we have mentioned the stability subgroup is G_2. In addition G_2 acts transitively on the S^6 in $\Lambda^1_7(\mathbb{R}^7)$ with stability subgroup $SU(3)$. Because of this, the component of ϵ_2 in $\Lambda^1_7(\mathbb{R}^7)$ can be set along the direction $i(1-e_{1234})$. However, since the stability subgroup is $SU(3)$, there are two more additional singlets. As a result, this case applies to $N=4$ backgrounds which we shall investigate below.

3.3 Three spinors

To find the Killing spinors of $N=3$ backgrounds, we assume that we have selected the first two Killing spinors as it has been described above. Therefore, we have to consider two cases. The first case is when the first two Killing spinors ϵ_1, ϵ_2 are the $SU(4) \ltimes \mathbb{R}^8$ invariant spinors (3.11). The decomposition of Δ^+_{16} under $SU(4)$ is

$$\Delta^+_{16} = \mathbb{R} < a_1(1+e_{1234}) > \oplus \mathbb{R} < a_2 i(1-e_{1234}) > \oplus \Lambda^2_6(\mathbb{C}^4) \oplus \Lambda^1_4(\mathbb{C}^4) \oplus \Lambda^1_4(\mathbb{C}^4)\ . \tag{3.13}$$

The third spinor ϵ_3 must be linearly independent from both $a_1(1+e_{1234})$ and $a_2 i(1-e_{1234})$. Suppose that $\epsilon_3 \in \Lambda^2_6(\mathbb{C}^4)$. It is known that the generic orbit of $SU(4)$ in $\Lambda^2_6(\mathbb{C}^4)$ can be represented by $\mu_1 e_{12} + \mu_2 e_{34}$, $\mu_1 \neq \pm\mu_2$ and has stability subgroup $SU(2)\times SU(2)$. However, there are at least two more real spinors invariant under the $SU(2)\times SU(2)$ subgroup of $SU(4)$ and so this case is suitable for backgrounds with $N>3$. However, it is well-known that there is a special orbit of $SU(4)$ in $\Lambda^2_6(\mathbb{C}^4)\oplus\Lambda^1_4$ represented by the real spinor $e_{12}-e_{34}$ which has enhanced stability subgroup $Sp(2)$. In addition, decomposing Δ_8 under $Sp(2)$ which can be done using $\mathfrak{sp}(2) = \mathfrak{so}(5)$, one can find that there are no additional singlets. To summarize, the Killing spinors of $N=3$ backgrounds are

$$\begin{aligned} \epsilon_1 &= f(1+e_{1234})\ , \\ \epsilon_2 &= g_1(1+e_{1234}) + i g_2(1-e_{1234})\ , \\ \epsilon_3 &= h_1(1+e_{1234}) + i h_2(1-e_{1234}) + h_3(e_{12}-e_{34})\ , \end{aligned} \tag{3.14}$$

with stability subgroup $Sp(2)\ltimes\mathbb{R}^8$ in $Spin(9,1)$. One can continue to investigate whether there are other cases of $N=3$ backgrounds. It turns out that there are no other possibilities.

3.4 Four spinors

Continuing in the same way as in the above cases, one can show that there are two cases to consider with four spinors. One case has stability subgroup $SU(3)$ and the other has stability subgroup $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$. A basis in the space of singlets in the former case is

$$\begin{aligned} \eta_1 &= 1 + e_{1234} \ , \quad \eta_2 = i(1 - e_{1234}) \ , \\ \eta_3 &= e_{15} + e_{2345} \ , \quad \eta_4 = i(e_{15} - e_{2345}) \end{aligned} \tag{3.15}$$

and a basis of singlets in the latter case is

$$\begin{aligned} \eta_1 &= 1 + e_{1234} \ , \quad \eta_2 = i(1 - e_{1234}) \ , \\ \eta_3 &= e_{12} - e_{34} \ , \quad \eta_4 = i(e_{12} + e_{34}) \, . \end{aligned} \tag{3.16}$$

The Killing spinors of supersymmetric backgrounds are linear combinations of the (constant) spinors in the above bases. However, we shall argue that in the case of heterotic string, one can always find a gauge such that the Killing spinors are constant and can be identified with the bases elements above.

3.5 Eight spinors

Similarly, there are two stability subgroups in $Spin(9,1)$ that leave invariant eight spinors. One stability subgroup is $SU(2)$ and a basis in the space of singlets is

$$\begin{aligned} \eta_1 &= 1 + e_{1234} \ , & \eta_2 &= i(1 - e_{1234}) \ , \\ \eta_3 &= e_{12} - e_{34} \ , & \eta_4 &= i(e_{12} + e_{34}) \ , \\ \eta_5 &= e_{15} + e_{2345} \ , & \eta_6 &= i(e_{15} - e_{2345}) \ , \\ \eta_7 &= e_{52} + e_{1345} \ , & \eta_8 &= i(e_{52} - e_{1345}) \, . \end{aligned} \tag{3.17}$$

The other stability subgroup is $\mathbb{R}^8$ and a basis in the space of singlets is

$$\begin{aligned} \eta_1 &= 1 + e_{1234} \ , & \eta_2 &= i(1 - e_{1234}) \ , \\ \eta_3 &= e_{12} - e_{34} \ , & \eta_4 &= i(e_{12} + e_{34}) \ , \\ \eta_5 &= e_{13} + e_{24} \ , & \eta_6 &= i(e_{13} - e_{24}) \ , \\ \eta_7 &= e_{23} - e_{14} \ , & \eta_8 &= i(e_{23} + e_{14}) \, . \end{aligned} \tag{3.18}$$

The Killing spinors of supersymmetric backgrounds with eight supersymmetries are again linear combinations of the (constant) spinors in the above bases. As in the previous case of four Killing spinors, it can always be arranged such that the Killing spinors are identified with the bases elements above.

Some of the results presented in this section are summarized in the table 1.

4. Parallel spinors and forms

4.1 Holonomy, gauge symmetry and Killing spinors

As we have mentioned, the gravitino Killing spinor equation of heterotic strings is a parallel transport equation for a metric connection with skew-symmetric torsion $\hat{\nabla}$. Therefore, the

G	N = 1	N = 2	N = 3	N = 4	N = 8	N = 16
$Spin(7) \ltimes \mathbb{R}^8$	$\surd$	-	-	-	-	-
$SU(4) \ltimes \mathbb{R}^8$	-	$\surd$	-	-	-	-
G_2	-	$\surd$	-	-	-	-
$Sp(2) \ltimes \mathbb{R}^8$	-	-	$\surd$	-	-	-
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	-	-	-	$\surd$	-	-
$SU(3)$	-	-	-	$\surd$	-	-
$\mathbb{R}^8$	-	-	-	-	$\surd$	-
$SU(2)$	-	-	-	-	$\surd$	-
$\{1\}$	-	-	-	-	-	$\surd$

Table 1: N denotes the number of parallel spinors and G their stability subgroup in $Spin(9,1)$. $\surd$ denotes the cases for which the parallel spinors occur. $-$ denotes the cases that do not occur.

main tool to investigate the existence of solutions of such an equation is the holonomy of $\hat{\nabla}$, $\mathrm{hol}(\hat{\nabla})$. The bundle of parallel spinors $\hat{\mathcal{K}}$ is spanned by the singlets of the decomposition of the Majorana-Weyl representation $\Delta^+_{\mathbf{16}}$ under $\mathrm{hol}(\hat{\nabla})$. In particular, we have

$$0 \to \hat{\mathcal{K}} \to \mathcal{S} \to \mathcal{S}/\hat{\mathcal{K}} \to 0 \ , \tag{4.1}$$

where $\mathcal{S}$ is the associated bundle of the principal spin bundle with typical fibre $\Delta^+_{\mathbf{16}}$. The vector bundle $\hat{\mathcal{K}}$ is topologically trivial and so it is equipped with the trivial connection ∂. In particular, one can introduce a basis ($\eta_i, i = 1, \dots, \mathrm{rank}\,\hat{\mathcal{K}}$) of constant spinors in $\hat{\mathcal{K}}$.

The Killing spinor equations of the heterotic string are covariant under (local) $Spin(9,1)$ gauge transformations as those of IIB supergravity. However, unlike the cases of IIB and eleven-dimensional supergravities, the Lie algebra $\mathfrak{spin}(9,1)$ of the gauge group of the Killing spinor equations *coincides* with the Lie algebra that the (super)covariant derivative $\hat{\nabla}$ takes values in. This in particular implies that the restriction of $\hat{\nabla}$ on the sections of $\hat{\mathcal{K}}$ can be trivialized with $Spin(9,1)$ local gauge transformation. As a result, there is a gauge, up to local $Spin(9,1)$ transformations, such that the *parallel spinors* of $\hat{\nabla}$ are *constant* and so they can be identified with a constant basis η_i. Of course the basis η_i is defined up to a (constant) general linear transformation $GL(\mathrm{rank}\,\hat{\mathcal{K}}, \mathbb{R})$. This transformation can be used to simplify the expressions for the Killing spinors, for more details see [39]. We remark that in IIB and eleven-dimensional supergravities, *there is not always a choice of a gauge for which the solutions of the gravitino Killing spinor equations are constant.* For example, one can adapt the results of [34] to show that the only maximally supersymmetric background of IIB and eleven-dimensional supergravities with constant Killing spinors is locally isometric to Minkowski spacetime.

Given a constant basis η_i of parallel spinors in $\hat{\mathcal{K}}$, the most general Killing spinors can be written as

$$\epsilon_r = \sum_i f_{ri}\eta_i \ , \qquad r = 1, \dots, N \ , \quad i = 1, \dots, \mathrm{rank}\ \mathcal{K} \ , \tag{4.2}$$

where $f = (f_{ri})$ is a real constant matrix. In general $N <$ rank $\hat{\mathcal{K}}$ because some parallel spinors may not solve the dilatino or gaugino Killing spinor equations. The matrix f can be thought of as the inclusion of the bundle of Killing spinors $\mathcal{K}$ in $\hat{\mathcal{K}}$.

To investigate all the supersymmetric backgrounds of the heterotic string, one has to determine the cases for which $N <$ rank $\hat{\mathcal{K}}$ for $N > 1$. It is well-known that there are such backgrounds as for example the group manifolds that have been mentioned in the introduction. In what follows, we shall only consider the cases for which $N =$ rank $\hat{\mathcal{K}}$. These are the so called maximally supersymmetric G-backgrounds in the terminology of [31]. In the first few cases, we shall allow the coefficients f in (4.2) to be spacetime functions and show that the parallel transport equations imply that f can be taken to be the identity, up to a local $Spin(9,1)$ and constant $GL(N,\mathbb{R})$ transformations, in agreement with the general argument presented above.

We have mentioned in the introduction that there are null and timelike supersymmetric backgrounds. These can be distinguished by the properties of their Killing spinors ($N =$ rank $\hat{\mathcal{K}}$). The Killing spinors of null supersymmetric backgrounds satisfy $\Gamma^-\epsilon = 0$, (see appendix A for our spinor conventions). Since $\epsilon \in \Delta^+_{\mathbf{16}}$, this condition implies that the Killing spinors are also chiral with respect to the Clifford subalgebra Cliff$(\mathbb{R}^8)$ of Cliff$(\mathbb{R}^{9,1})$, where $\mathbb{R}^8 = \mathbb{R} < e^1, \dots, e^4, e^6, \dots, e^9 >$. There are null supersymmetric backgrounds with even and odd number of Killing spinors.

The timelike supersymmetric backgrounds admit always even number of Killing spinors. Half of these spinors satisfy the condition $\Gamma^-\epsilon = 0$ while the other half satisfies the condition $\Gamma^+\epsilon = 0$. Therefore the Killing spinors do not have a definite chirally with respect to the above Cliff$(\mathbb{R}^8)$ subalgebra.

4.2 Parallel forms

It has been known for some time that an alternative way to characterize the geometry of supersymmetric heterotic backgrounds is in terms of the spacetime form bilinears of the parallel spinors, see e.g. [5, 14, 7]. In the heterotic case, a consequence of the Killing spinor equations is that all the spacetime form bilinears of the parallel spinors are also *parallel* with respect to the connection $\hat{\nabla}$. This is because $\hat{\nabla}$ is a connection that takes values in $\mathfrak{spin}(9,1)$ and so preserves the gamma-matrices and the spinor inner product, i.e. $\hat{\nabla}\epsilon_r = 0$, $r = 1, \dots, N$, implies that

$$\hat{\nabla}\alpha_{rs} = 0 \ , \quad r, s = 1, \dots, N \ , \tag{4.3}$$

where α_{rs} represents all the form spinor bilinears, see appendix A for the definition of α.

A converse to the above statement has been presented in [40]. In the case of the heterotic string a stronger statement is valid. In particular, if the forms α_{rs} are spinor bilinears of some spinors ϵ_r and $\hat{\nabla}\alpha_{rs} = 0$, then $\hat{\nabla}\epsilon_r = 0$. This is because the stability subgroups of the parallel spinors can also be characterized as those subgroups of $Spin(9,1)$ that leave the forms α_{rs} invariant. Therefore, one can use the form spinor bilinears to give an alternative description of the geometry of spacetime of supersymmetric backgrounds.

The parallel forms of supersymmetric backgrounds generate a ring under the wedge product. It turns out that the ring of null supersymmetric backgrounds is nilpotent, i.e. the wedge product of any two forms in the ring vanishes. In all cases, there is a null parallel one-form $\kappa = \mathrm{e}^-$ and all the rest of the generators of the ring are of the form

$$\alpha = \mathrm{e}^- \wedge \phi \ , \tag{4.4}$$

where $(\mathrm{e}^+, \mathrm{e}^-, e^i)$ is a light-cone frame adapted to the metric. Although $\hat\nabla\alpha = 0$, the form ϕ is not parallel with respect to the $\hat\nabla$ connection.[4] In particular, we have

$$\begin{aligned} &\hat\nabla_A \phi_{i_1\dots i_k} = 0 \ , \quad \hat\nabla_A \phi_{B_1\dots B_{k-1}+} = 0 \ , \\ &\hat\nabla_A \phi_{i_1\dots i_{k-1}-} = \hat\Omega_{A,}{}^m{}_- \phi_{i_1\dots i_{k-1}m} \ . \end{aligned} \tag{4.5}$$

Nevertheless in many cases it is convenient to use the form ϕ to describe the geometry of spacetime.

The timelike supersymmetric backgrounds admit *at least* three parallel one forms $\kappa = \mathrm{e}^-$, $\kappa' = \mathrm{e}^+$ and $\hat\kappa = e^1$. The $N > 2$ backgrounds admit more than three parallel one-forms. The associated ring of parallel forms is not nilpotent. At the end of this section, we give the generators of the rings of the parallel forms of all the supersymmetric backgrounds.

Some geometric properties of the spacetime follow immediately from (4.3). For example, let κ be a one-form parallel spinor bilinear. Then (4.3) implies that κ is parallel, $\hat\nabla\kappa = 0$. The associated vector field X with respect to the spacetime metric is also parallel, $\hat\nabla X = 0$. A consequence of this is that

$$\begin{aligned} \mathcal{L}_X g &= 0 \\ d\kappa &= i_X H \ , \end{aligned} \tag{4.6}$$

i.e. X is Killing and that the rotation of κ is equal to the particular component of the flux H. In addition, if H satisfies the Bianchi identity, which it does at the lowest order[5] in α', then

$$\mathcal{L}_X H = d i_X H + i_X dH = d i_X H = 0 \ , \tag{4.7}$$

and so H is also invariant under the one-parameter family of diffeomorphisms generated by X.

Next suppose that X, Y are $\hat\nabla$-parallel vector fields and denote with κ_X and κ_Y the associated one-forms. The commutator of such two Killing vector fields is Killing because $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$. In addition, it is known that $i_{[X,Y]} = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X$ and so

$$i_{[X,Y]} H = \mathcal{L}_X i_Y H = \mathcal{L}_X d\kappa_Y = d\mathcal{L}_X \kappa_Y = d\kappa_{[X,Y]} \ . \tag{4.8}$$

Therefore, the commutator $[X, Y]$ is also parallel with respect $\hat\nabla$. However $\kappa_{[X,Y]}$ *may not* be associated with a one-form parallel spinor bilinear.

[4] However, it is parallel with respect to another connection which takes values in the compact subalgebra of the holonomy group of the null supersymmetric backgrounds.

[5] It is expected that H will remain invariant after all perturbative corrections in α' are taken into account provided that the classical background is invariant under the transformations generated by X. This is because the corrections are polynomials of the Riemann curvature R, F, H and their covariant derivatives which are invariant under X.

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Another aspect of the form spinor bilinears that arise in the context of supersymmetric heterotic string backgrounds is whether or not they are invariant under the Killing vectors of these backgrounds. Let X be a Killing vector associated with a one-form spinor bilinear κ and α be a k-form spinor bilinear. Using $\hat{\nabla}\alpha = \hat{\nabla}\kappa = 0$, one can show that

$$(\mathcal{L}_X\alpha)_{A_1\dots A_k} = k(-1)^k(i_X H)^B{}_{[A_1}\alpha_{A_2\dots A_k]B} \ , \tag{4.9}$$

where $A_1, \dots, A_k, B = -, +, i$. Therefore $\mathcal{L}_X\alpha = 0$, iff the rotation of X, $i_X H$, leaves invariant the form α. We shall find that the dilatino Killing spinor equation implies such conditions.

It also turns out that the geometry of the spacetime of supersymmetric backgrounds can be described using a minimal set of parallel forms. This particularly applies to the conditions that arise from the gravitino Killing spinor equation. This is similar to the characterization of Kähler manifolds as the Riemannian manifolds that admit a parallel almost complex structure. The generators of the ring of parallel forms or the rings themselves for the supersymmetric backgrounds, up to Hodge duality, are summarized in table 4.2 below.

5. $N = 1$ backgrounds

5.1 Supersymmetry conditions

In section 3, we have shown that the Killing spinor can be chosen as $\epsilon = f(1+e_{1234})$ and has stability subgroup $Spin(7) \ltimes \mathbb{R}^8$, where f is a real function of the spacetime. Substituting this into the gravitino Killing spinor equation, we find

$$\begin{aligned}&\partial_A \log f(1+e_{1234}) - \frac{1}{8}\hat{\Omega}_{A,\gamma\delta}\epsilon^{\gamma\delta}{}_{\bar{\alpha}\bar{\beta}}\Gamma^{\bar{\alpha}\bar{\beta}}1 + \frac{1}{4}\hat{\Omega}_{A,\bar{\alpha}\bar{\beta}}\Gamma^{\bar{\alpha}\bar{\beta}}1 + \frac{1}{2}\hat{\Omega}_{A,\alpha}{}^{\alpha}1\\ &-\frac{1}{2}\hat{\Omega}_{A,\alpha}{}^{\alpha}e_{1234} + \frac{1}{2}\hat{\Omega}_{A,+\alpha}\Gamma^{+\alpha}e_{1234} + \frac{1}{2}\hat{\Omega}_{A,+\bar{\alpha}}\Gamma^{+\bar{\alpha}}1 + \frac{1}{2}\hat{\Omega}_{A,-+}(1+e_{1234}) = 0\,. \end{aligned}\tag{5.1}$$

The above equation can be expanded in the basis (A.9). Setting every component in this basis to zero, we find the conditions

$$\partial_A \log f + \frac{1}{2}\hat{\Omega}_{A,-+} = 0 \ , \tag{5.2}$$

$$\hat{\Omega}_{A,\alpha}{}^{\alpha} = 0 \ , \qquad \hat{\Omega}_{A,\bar{\alpha}\bar{\beta}} - \frac{1}{2}\hat{\Omega}_{A,\gamma\delta}\epsilon^{\gamma\delta}{}_{\bar{\alpha}\bar{\beta}} = 0 \ , \tag{5.3}$$

$$\hat{\Omega}_{A,+\bar{\alpha}} = \hat{\Omega}_{A,+\alpha} = 0\,. \tag{5.4}$$

The components $\hat{\Omega}_{A,-\alpha}$ and $\hat{\Omega}_{A,-\bar{\alpha}}$ are unconstrained.

Similarly, one substitutes $\epsilon = f(1+e_{1234})$ into the dilatino Killing spinor equation to find

$$\left(\Gamma^A\partial_A\Phi - \frac{1}{12}\Gamma^{ABC}H_{ABC})(1+e_{1234}\right) = 0\,. \tag{5.5}$$

Expanding this in the basis (A.9), we get that

$$\partial_{\bar{\alpha}}\Phi + \frac{1}{6}H_{\beta_1\beta_2\beta_3}\epsilon^{\beta_1\beta_2\beta_3}{}_{\bar{\alpha}} - \frac{1}{2}H_{\bar{\alpha}\beta}{}^{\beta} - \frac{1}{2}H_{-+\bar{\alpha}} = 0 \ , \tag{5.6}$$

Supersymmetry	Killing Vectors	Parallel Forms
$N=1\ \ Spin(7)\ltimes\mathbb{R}^8$	1	$\mathrm{e}^-, \mathrm{e}^-\wedge\phi$
$N=2\ \ SU(4)\ltimes\mathbb{R}^8$	1	$\mathrm{e}^-\ , \mathrm{e}^-\wedge\chi\ , \mathrm{e}^-\wedge\omega$
$N=2\ \ G_2$	3	$\mathrm{e}^-\ ,\ \mathrm{e}^+\ ,\ e^1\ ,\ \varphi$
$N=3\ \ Sp(2)\ltimes\mathbb{R}^8$	1	$\mathrm{e}^-\ , \mathrm{e}^-\wedge\omega_I\ , \mathrm{e}^-\wedge\omega_J\ , \mathrm{e}^-\wedge\omega_K$
$N=4\ \ (SU(2)\times SU(2))\ltimes\mathbb{R}^8$	1	$\mathrm{e}^-\ , -\mathrm{e}^-\wedge(e^1\wedge e^6+e^2\wedge e^7)$
		$-\mathrm{e}^-\wedge(e^3\wedge e^8+e^4\wedge e^9)$
		$\mathrm{e}^-\wedge(e^1+ie^6)\wedge(e^2+ie^7)$
		$\mathrm{e}^-\wedge(e^3+ie^8)\wedge(e^4+ie^9)$
$N=4\ \ SU(3)$	4	$\mathrm{e}^-\ , \mathrm{e}^+\ ,\ e^1\ ,\ e^6\ ,\ \hat{\omega}\ ,\ \hat{\chi}$
$N=8\ \ \mathbb{R}^8$	1	$\mathrm{e}^-\wedge\psi\,,\quad \psi\in\Lambda^{\mathrm{ev}+}(\mathbb{R}^8)$
$N=8\ \ SU(2)$	6	$\mathrm{e}^-\ ,\ \mathrm{e}^+\ ,\ e^1\ ,\ e^6\ ,\ e^2\ ,\ e^7\ ,$
		$-e^3\wedge e^8-e^4\wedge e^9\ ,\ (e^3+ie^8)\wedge(e^4+ie^9)$
$N=16\ \ \{1\}$	10	$e^A\ ,\quad A=0,\dots,9$

Table 2: The first column gives the number of Killing vectors that are constructed from Killing spinor bilinears of a supersymmetric background. The second column gives a minimal set of $\hat{\nabla}$-parallel forms which characterizes the geometry of the supersymmetric background, where $\Lambda^{\mathrm{even}+}(\mathbb{R}^8)=\Lambda^0(\mathbb{R}^8)\oplus\Lambda^2(\mathbb{R}^8)\oplus\Lambda^{4+}(\mathbb{R}^8)$ and $\Lambda^{4+}(\mathbb{R}^8)$ is the space of self-dual four-forms in $\mathbb{R}^8$, and

$$
\begin{aligned}
&\chi=(e^1+ie^6)\wedge(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\ ,\\
&\omega=-e^1\wedge e^6-e^2\wedge e^7-e^3\wedge e^8-e^4\wedge e^9\ ,\quad \phi=\mathrm{Re}\,\chi-\frac{1}{2}\omega\wedge\omega\ ,\\
&\hat{\omega}=-e^2\wedge e^7-e^3\wedge e^8-e^4\wedge e^9\ ,\quad \hat{\chi}=(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\ ,\\
&\varphi=\mathrm{Re}\hat{\chi}+e^6\wedge\hat{\omega}\ ,\quad \omega_I=\omega\ ,\\
&\omega_J=\mathrm{Re}[(e^1+ie^6)\wedge(e^2+ie^7)]+(e^3+ie^8)\wedge(e^4+ie^9)]\ ,\\
&\omega_K=-\mathrm{Im}[(e^1+ie^6)\wedge(e^2+ie^7)+(e^3+ie^8)\wedge(e^4+ie^9)]\ .
\end{aligned}
$$

$$\partial_+\Phi=0\ , \tag{5.7}$$

$$H_{+\alpha}{}^{\alpha}=0\ ,\qquad -H_{+\bar{\alpha}_1\bar{\alpha}_2}+\frac{1}{2}H_{+\beta_1\beta_2}\epsilon^{\beta_1\beta_2}{}_{\bar{\alpha}_1\bar{\alpha}_2}=0\,. \tag{5.8}$$

The components $\partial_-\Phi$ and H_{-ij} remain undetermined by the dilatino Killing spinor equation, where $i=\alpha,\bar{\alpha}$ and similarly j.

5.2 The geometry of spacetime

5.2.1 The holonomy of $\hat{\nabla}$ and supersymmetry

The gravitino Killing spinor equation implies that the holonomy of the $\hat{\nabla}$ connection is contained in $Spin(7)\ltimes\mathbb{R}^8$. This may have been expected on general grounds because the Killing spinor ϵ is parallel with respect to $\hat{\nabla}$ and so the holonomy of $\hat{\nabla}$ should be contained in the stability subgroup of the Killing spinor ϵ in $Spin(9,1)$.

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One can also see this explicitly in the gauge $f=1$. This gauge can be attained by the spinorial transformation $e^{b\Gamma^{05}}$, which induces a Lorentz gauge transformation on $\hat{\nabla}$ and a Lorentz rotation on the fluxes. The action of $e^{b\Gamma^{05}}$ on the Killing spinor ϵ is to scale it with e^b. Therefore setting $b=-\log|f|$, the spacetime dependence of the Killing spinor can be gauged away and so the Killing spinor can be written as $\epsilon = 1+e_{1234}$. In this gauge

$$\hat{\Omega}_{A,+-} = 0 \ , \tag{5.9}$$

which together with (5.4) imply that all the components of $\hat{\Omega}_{A,+B}=0$. It is then easy to see that the remaining components of the connection one-form, $\hat{\Omega}=\hat{\Omega}_A e^A$, take values in $\mathfrak{spin}(7)\oplus_s \mathbb{R}^8$. Note however that for generic $N=1$ backgrounds, the Levi-Civita connection *does not* have $Spin(7)\ltimes\mathbb{R}^8$ holonomy.

The converse is also valid. If $\mathrm{hol}(\hat{\nabla})\subseteq Spin(7)\ltimes\mathbb{R}^8$, there is a spinor ϵ which is parallel with respect to $\hat{\nabla}$ and so ϵ satisfies the gravitino Killing spinor equation. Thus the existence of a solution for the gravitino Killing spinor equation can be entirely characterized by the holonomy of $\hat{\nabla}$.

To investigate further the geometry of spacetime, it is convenient to introduce the $\hat{\nabla}$-parallel forms associated with the parallel spinor bilinears. It turns out that most of the fluxes and geometry can be expressed in terms of these bilinears.

5.2.2 Spacetime forms

Using the formulae that we have collected in appendix A, one can find that the non-vanishing Killing spinor bilinears[6] are a one-form

$$\kappa=\kappa(\epsilon,\epsilon)=f^2(e^0-e^5) \ , \tag{5.10}$$

and a five-form

$$\tau=\tau(\epsilon,\epsilon)=f^2(e^0-e^5)\wedge\phi \ , \tag{5.11}$$

where

$$\phi=\mathrm{Re}\chi-\frac{1}{2}\omega\wedge\omega \ , \tag{5.12}$$

and χ and ω are defined in appendix A, see also table 4.2. It is easy to recognize that ϕ is the usual $Spin(7)$-invariant four-form on eight-dimensional manifolds. The forms ω and χ are not individually well-defined on the spacetime.

To proceed, we introduce a frame $\mathrm{e}^+,\mathrm{e}^-,\mathrm{e}^\alpha,\mathrm{e}^{\bar\alpha}$, where $\mathrm{e}^-=(1/\sqrt{2})\,(-e^0+e^5)$, $\mathrm{e}^+=(1/\sqrt{2})\,(e^0+e^5)$, and $\mathrm{e}^\alpha=(1/\sqrt{2})\,(e^\alpha+ie^{\alpha+5}), \mathrm{e}^{\bar\alpha}=(1/\sqrt{2})\,(e^\alpha-ie^{\alpha+5})$, and $(e^0,\dots,e^9)$ is the orthonormal frame in appendix A. The spacetime metric can be rewritten as

$$ds^2=2\mathrm{e}^+\mathrm{e}^-+2\delta_{\alpha\bar\beta}\mathrm{e}^\alpha\mathrm{e}^{\bar\beta} \, . \tag{5.13}$$

In this new frame[7] $\kappa=f^2\,\mathrm{e}^-$ and $\tau=f^2\,\mathrm{e}^-\wedge\phi$. Therefore the ring of form spinor bilinears under the wedge product is nilpotent, i.e. the wedge product of any two forms vanishes.

[6]We have normalized the Killing spinor ϵ with an additional factor of $1/\sqrt{2}$.

[7]We have normalized the forms with a further factor of $1/\sqrt{2}$.

As we have explained in section 4.2, κ and τ are $\hat{\nabla}$-parallel. Therefore the vector field $X = f^2\, \mathrm{e}_+$ associated with the one-form κ with respect to the spacetime metric is also $\hat{\nabla}$-parallel, i.e. $\hat{\nabla}\, X = 0$, where $e^A(e_B) = \delta^A_B$ and e_B is the co-frame. This in turn implies that X is Killing and $d\kappa = i_X H$. Consequently, $di_X H = 0$ and so the Bianchi identity, $dH = 0$, implies that $\mathcal{L}_X H = 0$. The three-form field strength H is invariant under the isometries generated by X. In addition (5.7) implies that $\mathcal{L}_X \Phi = 0$ as well. Therefore the metric and both fluxes H and Φ are invariant under X. Furthermore as we shall explain in detail in the next section, (5.8) implies that H_{+AB} takes values in $\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$. Using (4.9), one finds that

$$\mathcal{L}_X \tau = 0\,. \tag{5.14}$$

Therefore, the parallel vector field X *leaves invariant* the $Spin(7) \ltimes \mathbb{R}^8$-structure of spacetime. It turns out that this is a generic property of all null supersymmetric heterotic string backgrounds that we investigate. The null parallel vector field preserves the $K \ltimes \mathbb{R}^8$-structure of the spacetime.

5.2.3 The solution of the Killing spinor equations

To further investigate the Killing spinor equations, we decompose the space of two-, three- and four-forms under $Spin(7)$ as as $\Lambda^2(\mathbb{R}^8) = \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{21}}$, $\Lambda^3(\mathbb{R}^8) = \Lambda^3_{\mathbf{8}} \oplus \Lambda^3_{\mathbf{48}}$, $\Lambda^4(\mathbb{R}^8) = \Lambda^4_+(\mathbb{R}^8) \oplus \Lambda^4_-(\mathbb{R}^8)$, $\Lambda^4_+ = \Lambda^4_{\mathbf{1}} \oplus \Lambda^4_{\mathbf{7}} \oplus \Lambda^4_{\mathbf{27}}$ and $\Lambda^4_- = \Lambda^4_{\mathbf{35}}$, where

$$\begin{aligned}
&\Lambda^2_{\mathbf{7}} = \{\alpha \in \Lambda^2(\mathbb{R}^8)|\, *(\alpha \wedge \phi) = -3\alpha\}\ , \quad \Lambda^2_{\mathbf{21}} = \{\alpha \in \Lambda^2(\mathbb{R}^8)|\, *(\alpha \wedge \phi) = \alpha\}\\
&\Lambda^3_{\mathbf{8}} = \{*(\alpha \wedge \phi)|\, \alpha \in \Lambda^1(\mathbb{R}^8)\}\ , \quad \Lambda^3_{\mathbf{48}} = \{\alpha \in \Lambda^3(\mathbb{R}^8)|\, \alpha \wedge \phi = 0\}\ ,\\
&\Lambda^4_{\mathbf{1}} = \{r\, \phi|\, r \in \mathbb{R}\}\,.
\end{aligned} \tag{5.15}$$

The representation $\Lambda^2_{\mathbf{21}}$ can be identified with the adjoint representation of $\mathfrak{spin}(7)$, so $\mathfrak{spin}(7) = \mathfrak{so}(7) = \Lambda^2_{\mathbf{21}}$. Using the above decompositions, the conditions that arise from the gravitino Killing spinor equation (5.6)-(5.8) in the gauge $f = 1$ can be written as

$$\hat{\Omega}_{A,+B} = 0\ , \quad \hat{\Omega}^{\mathbf{7}}_{A,ij} = 0\ , \tag{5.16}$$

where the projection to the seven-dimensional representation is done in the indices $i, j = 1, \ldots 4, 6, \ldots, 9$. In addition, the conditions that arise from the dilatino Killing spinor equation can be rewritten as

$$\partial_i \Phi + \frac{1}{12} H_{jkl} \phi^{jkl}{}_i - \frac{1}{2} H_{-+i} = 0\ , \quad \partial_+ \Phi = 0\ , \quad H^{\mathbf{7}}_{+ij} = 0. \tag{5.17}$$

The conditions (5.16) and (5.17) can be solved to determine most of the components of the flux in terms of the geometry. In particular, the first equation in (5.16) implies that $\kappa = \mathrm{e}^-$ is parallel and so $i_X H = d\kappa = d\mathrm{e}^-$. The second equation is equivalent to $\hat{\nabla}_A \phi_{ijkl} = 0$ and so in particular implies that

$$\begin{aligned}
&\hat{\nabla}_- \phi_{ijkl} = 0\ ,\\
&\hat{\nabla}_m \phi_{ijkl} = 0\,.
\end{aligned} \tag{5.18}$$

These equations can be solved for the fluxes to give

$$
\begin{aligned}
H^{\mathbf{7}}_{-ij} &= H_{-ij} - \frac{1}{2} H_{-kl}\phi^{kl}{}_{ij} = -\frac{1}{12}\phi^{pqr}{}_i \nabla_- \phi_{pqrj} \;, \\
H_{ijk} &= -\frac{1}{4!}\nabla_{[m_1}\phi_{m_2\dots m_5]}\epsilon^{m_1 m_2 \dots m_5}{}_{ijk} + \theta^m \phi_{mijk} \;,
\end{aligned} \tag{5.19}
$$

where

$$
\theta_i = -\frac{1}{36}\nabla^p \phi_{p j_1 j_2 j_3}\phi^{j_1 j_2 j_3}{}_i \,. \tag{5.20}
$$

Observe that θ is analogous to the Lee form of eight-dimensional Riemannian manifolds with a $Spin(7)$-structure. To derive the second equation in (5.19), we have use the results of [24]. In addition the first condition in (5.17) implies that the $H^{\mathbf{8}}_{ijk}$ component of H which is determined by θ can be expressed in terms of the derivative of the dilaton and the $H_{+-i} = (d\kappa)_{-i}$ component of the flux. If $H_{+-i} \neq \partial_i f$, then the spacetime is not conformally balanced.

Therefore the metric and three-form flux of the supersymmetric spacetime can be written as

$$
\begin{aligned}
ds^2 &= 2\mathrm{e}^+\mathrm{e}^- + \delta_{\alpha\bar\beta}\mathrm{e}^\alpha \mathrm{e}^{\bar\beta} \\
H &= \mathrm{e}^+ \wedge d\mathrm{e}^- + \Omega^{\mathbf{7}}_{-,ij}\; \mathrm{e}^- \wedge e^i \wedge e^j + \frac{1}{2} H^{\mathbf{21}}_{-ij}\mathrm{e}^- \wedge e^i \wedge e^j \\
&\qquad + \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k \;,
\end{aligned} \tag{5.21}
$$

where H_{ijk} is given in (5.19). The component $H^{\mathbf{21}}_{-ij}$ of the fluxes is not determined by the Killing spinor equations.

5.2.4 Local coordinates

One can introduce local coordinates on the spacetime M by adapting a coordinate u along the null Killing vector field X, $X = \frac{\partial}{\partial u}$. The spacetime metric can be written as

$$
ds^2 = 2U(dv + m_I dy^I)(du + V dv + n_I dy^I) + \gamma_{IJ} dy^I dy^J \;, \tag{5.22}
$$

where U, V, m_I, n_I and γ_{IJ} are functions of v, y^I coordinates, $I, J = 1, \dots, 8$. All the components of the metric are independent of u because X is Killing. In addition $U = f^2$. To see this, we adapt the frame

$$
\mathrm{e}^- = dv + m_I dy^I \;, \quad \mathrm{e}^+ = U(du + V dv + n_I dy^I) \;, \quad e^i = e^i_J dy^J \;, \tag{5.23}
$$

where $\gamma_{IJ} = \delta_{ij} e^i_I e^j_J$. The Killing vector field in this frame is

$$
X = f^2 e_+ = \frac{\partial}{\partial u} \;, \tag{5.24}
$$

where e_B is

$$
e_+ = U^{-1}\frac{\partial}{\partial u} \;, \quad e_- = \frac{\partial}{\partial v} - V\frac{\partial}{\partial u} \;,
$$

$$e_i = e_i^J \frac{\partial}{\partial y^J} + (-n_i + Vm_i)\frac{\partial}{\partial u} - m_i \frac{\partial}{\partial v} \ , \tag{5.25}$$

$e^i_I e^I_j = \delta^i{}_j$, $m_i = m_I e^I_i$ and $n_i = n_I e^I_i$. Using the above expression for the co-frame, the Killing vector field X can be written as

$$X = f^2 e_+ = f^2 U^{-1} \frac{\partial}{\partial u} = \frac{\partial}{\partial u} \, . \tag{5.26}$$

Therefore $U = f^2$. In particular, we can set $U = 1$ in the gauge $f = 1$.

A consequence of the torsion free condition for the Levi-Civita connection and $\hat{\Omega}_{A,+B} = 0$ is that

$$i_X H = dm \, . \tag{5.27}$$

So using $i_X H = d\kappa$, one finds that

$$d\kappa = dm \, . \tag{5.28}$$

As it may have been expected the off-diagonal part of the metric (5.22) proportional to m, which is responsible for the deviation from Penrose coordinates, is due to the rotation of the null geodesic congruence generated by κ. In addition (5.27) relates this term to the presence of non-vanishing H fluxes. Furthermore, the coordinate v of the spacetime can be specified by applying the Poincaré lemma on the closure relation $d(\kappa - m) = 0$.

5.2.5 A deformation family of $Spin(7)$-structures

The spacetime M of $N = 1$ supersymmetric heterotic string backgrounds can be interpreted as a two parameter Lorentzian deformation family[8] of an eight-dimensional manifold B with an $Spin(7)$-structure. To see this, observe that the metric (5.22) can be rewritten as

$$ds^2 = g_{ab} du^a du^b + g_{IJ}(dy^I + A^I_a du^a)(dy^J + A^J_b du^b) \ , \tag{5.29}$$

where

$$\begin{aligned} &g_{vu} + g_{IJ} A^I_v A^J_u = U \ , \quad g_{vv} + g_{IJ} A^I_v A^J_v = 2UV \ , \quad g_{uu} + g_{IJ} A^I_u A^J_u = 0 \\ &g_{JI} A^J_u = U m_I \ , \quad g_{JI} A^J_v = U n_I + UV m_I \ , \quad g_{IJ} = \gamma_{IJ} + 2U n_{(I} m_{J)} \ , \end{aligned} \tag{5.30}$$

and g_{ab}, g_{IJ}, A^I_a depend on all coordinates u^a, y^I, $((u^a) = (u, v))$. The components A^I_a can be thought of as the non-linear connection of the family.

The spacetime admits an integrable distribution of co-dimension eight. To see this, we adapt a frame

$$E^+ \ , \quad E^- \ , \quad E^i = \tilde{e}^i{}_J (dy^J + A^J_b du^b) \ , \tag{5.31}$$

to the metric (5.29), where E^+, E^- is a light-cone frame adapted to the two-dimensional part of the metric, $g_{ab} du^a du^b = 2E^- E^+$, and $\delta_{ij} \tilde{e}^i{}_I \tilde{e}^j{}_J = g_{IJ}$. Applying the Frobenius

[8] The family is trivial with respect to one of the two parameters.

theorem to the one-forms E^+, E^-, one can easily show that the spacetime is an integrable foliation of co-dimension eight with leaves the deformed manifold B given by $u, v =$ const.

It remains to determine the geometry of B that gets deformed. It is clear that B is a Riemannian manifold with metric $d\tilde{s}^2 = g_{IJ}dy^I dy^J$ equipped with a three-form $\tilde{H} = H|_B$. So one can construct a Riemannian connection $\hat{\tilde{\nabla}}$ on B with torsion $\tilde{H}$. In addition B admits a $Spin(7)$-invariant form $\tilde{\phi} = \phi|_B$. However these data are not compatible, i.e. in general $\hat{\tilde{\nabla}}\tilde{\phi} \neq 0$. To see this, observe that

$$e^i = \ell^i{}_j(E^i + p^j E^+ + q^i E^-) \tag{5.32}$$

for some non-vanishing p and q, $\det\ell \neq 0$, and similarly for the rest components of the frame. This in particular implies that the self-dual four form ϕ in the (E^i, E^+, E^-) frame has components in the E^+ and E^- directions. Taking the covariant derivative of ϕ, i.e. $\hat{\nabla}_i\phi$, one get contributions from $\hat{\nabla}_i E^+ = -\hat{\Omega}_i{}^+{}_j E^j$, and similarly from $\hat{\nabla}_i E^-$, which can be identified with the second fundamental form of B with respect to the connection $\hat{\nabla}$. Since these contributions do not apparently vanish, $\hat{\nabla}_i\phi = 0$, see (4.5), after restriction to B does not imply that $\hat{\tilde{\nabla}}_i\tilde{\phi} = 0$. Therefore B does not have a $Spin(7)$ structure compatible with the connection $\hat{\tilde{\nabla}}$. Nevertheless, B admits a $Spin(7)$-structure.

There is though a *special case* where the $Spin(7)$-structure of B is compatible with the $\hat{\tilde{\nabla}}$ connection. This is whenever the rotation of the null $\hat{\nabla}$-parallel vector field vanishes, $d\kappa = d\mathrm{e}^- = 0$, i.e. the metric is written in terms of Penrose coordinates. In this case, the $(\mathrm{e}^-, \mathrm{e}^+, e^i)$ and (E^-, E^+, E^i) frames are related as

$$\mathrm{e}^- = E^- \ , \quad \mathrm{e}^+ - n_i e^i = E^+ \ , \quad e^i = E^i - n^i E^- \, . \tag{5.33}$$

Then $\phi = \frac{1}{4!}\phi_{ijkl}e^i \wedge e^j \wedge e^k \wedge e^l$ can be written as $\phi = \psi + E^- \wedge \tau$, where $\psi = \frac{1}{4!}\phi_{ijkl}E^i \wedge E^j \wedge E^k \wedge E^l$. Thus we have

$$(\hat{\nabla}_i\phi)|_B = \hat{\tilde{\nabla}}_i\tilde{\phi} + (\hat{\nabla}_i E^-)|_B \wedge \tau|_B = \hat{\tilde{\nabla}}_i\tilde{\phi} = 0 \tag{5.34}$$

because $\nabla_i E^- = \nabla_i \mathrm{e}^- = -\Omega_{i,+A}E^A = 0$, since $\hat{\Omega}_{A,+B} = 0$, $E^+|_B = E^-|_B = 0$, and ϕ is parallel with respect to $\hat{\nabla}$ along the B directions, see (4.5), where $\tilde{\phi} = \phi|_B = \psi|_B$. In addition B is conformally balanced. This is because the dilatino Killing spinors equation (5.17) when restricted on B gives

$$\tilde{\theta}_i = 2\partial_i\tilde{\Phi} \ , \tag{5.35}$$

since the rotation of the vector field vanishes. Eight-dimensional Riemannian manifolds with a conformally balanced $Spin(7)$-structure compatible with a connection with skew-symmetric torsion have been investigated in [24]. Any eight-dimensional Riemannian manifold with a $Spin(7)$-structure admits a connection with skew-symmetric torsion

$$\tilde{H} = -\star d\tilde{\phi} + \star(\tilde{\theta} \wedge \tilde{\phi}) \ , \tag{5.36}$$

where the Lee form can also be written as $\tilde{\theta} = -\frac{1}{6}\star(\star d\phi \wedge \phi)$ and $\star$ is the Hodge duality operator of B for $d\mathrm{vol}(B) = \tilde{e}^1 \wedge \ldots \wedge \tilde{e}^4 \wedge \tilde{e}^6 \wedge \ldots \wedge \tilde{e}^9$. Note that $\tilde{e}^i = e^i|_B = E^i|_B$. Our form conventions are summarized in appendix A. Of course the torsion is required to satisfy the (generalized) Bianchi identity for applications to the heterotic string.

The geometry of B can also be given in terms of G-structures. It is known that there are four classes of $Spin(7)$-structures obtained by decomposing $\tilde\nabla\tilde\phi$ in terms of $Spin(7)$ representations [41]. These classes can be described in terms of the Lee form [42] as follows: W_0 ($d\tilde\phi = 0$), W_1 ($\tilde\theta = 0$), W_2 ($d\tilde\phi = \frac{6}{7}\,\tilde\theta \wedge \tilde\phi$) and $W = W_1 \oplus W_2$. The only restriction that we find on the $Spin(7)$-structure of B arising from supersymmetry is that it is conformally balanced, i.e. $\tilde\theta = 2d\tilde\Phi$. These geometries are in the same conformal class as those of the W_1 $Spin(7)$-structure. To see this, observe that under the conformal transformation $d\tilde s^2_\Omega = e^{2\Omega} d\tilde s^2$, the four-forms changes as $\tilde\phi^\Omega = e^{4\Omega}\tilde\phi$. Then the Lee form of $(d\tilde s^2_\Omega, \phi^\Omega)$ can be written in terms of the Lee form of $(d\tilde s^2, \tilde\phi)$ as $\tilde\theta^\Omega = \tilde\theta + \frac{14}{3} d\Omega$. Thus $\tilde\theta^\Omega = 0$ for $\Omega = -\frac{3}{7}\tilde\Phi$ and so $(d\tilde s^2_\Omega, \tilde\phi^\Omega)$, is in the W_1 class.

5.3 Field equations

It is straightforward to derive the field equations that follow as the integrability conditions of the Killing spinor equations. In this way, we find the minimal set of field equations that need to be solved in addition to solving the Killing spinor equations. For the case at hand, we find that the integrability conditions of the Killing spinor equations give

$$\begin{aligned}
E_{++} &= E_{+\alpha} = 0 \;, \\
E_{-+} &= -\frac{1}{2} e^{2\Phi} LH_{-+} \;, \\
E_{-\alpha} &= -\frac{1}{2} e^{2\Phi} LH_{-\alpha} + \frac{1}{2} BH_{-\alpha\gamma}{}^{\gamma} - \frac{1}{6}\epsilon_{\alpha}{}^{\bar\gamma_1\bar\gamma_2\bar\gamma_3} BH_{-\bar\gamma_1\bar\gamma_2\bar\gamma_3} \;, \\
E_{\alpha\beta} &= -\frac{1}{12}\epsilon_{\alpha}{}^{\bar\gamma_1\bar\gamma_2\bar\gamma_3} BH_{\beta\bar\gamma_1\bar\gamma_2\bar\gamma_3} - \frac{1}{12}\epsilon_{\beta}{}^{\bar\gamma_1\bar\gamma_2\bar\gamma_3} BH_{\alpha\bar\gamma_1\bar\gamma_2\bar\gamma_3} \;, \\
E_{\alpha\bar\beta} &= -\frac{1}{2} BH_{\alpha\bar\beta\gamma}{}^{\gamma} - \frac{1}{12}\epsilon_{\alpha}{}^{\bar\gamma_1\bar\gamma_2\bar\gamma_3} BH_{\bar\beta\bar\gamma_1\bar\gamma_2\bar\gamma_3} - \frac{1}{12}\epsilon_{\bar\beta}{}^{\gamma_1\gamma_2\gamma_3} BH_{\alpha\gamma_1\gamma_2\gamma_3} \;, \\
LH_{+\alpha} &= 0 \;, \\
e^{2\Phi} LH_{\alpha_1\alpha_2} &= -\frac{1}{2}\epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2} BH_{-+\bar\beta_1\bar\beta_2} \;, \\
e^{2\Phi} LH_{\alpha\bar\beta} &= -BH_{-+\alpha\bar\beta} + \frac{1}{6}\epsilon_{\alpha}{}^{\bar\gamma_1\bar\gamma_2\bar\gamma_3} BH_{\bar\beta\bar\gamma_1\bar\gamma_2\bar\gamma_3} - \frac{1}{6}\epsilon_{\bar\beta}{}^{\gamma_1\gamma_2\gamma_3} BH_{\alpha\gamma_1\gamma_2\gamma_3} \;, \\
LF_{+} &= 0 \;, \\
e^{2\Phi} LF_{\alpha} &= -BF_{-+\alpha} + BF_{\alpha\gamma}{}^{\gamma} - \frac{1}{3}\epsilon_{\alpha}{}^{\bar\gamma_1\bar\gamma_2\bar\gamma_3} BF_{\bar\gamma_1\bar\gamma_2\bar\gamma_3} \;, \\
L\Phi &= \frac{1}{2} e^{2\Phi} LH_{-+} + \frac{1}{4} BH_{\gamma}{}^{\gamma}{}_{\delta}{}^{\delta} + \frac{1}{24}\epsilon^{\bar\gamma_1\cdots\bar\gamma_4} BH_{\bar\gamma_1\cdots\bar\gamma_4} + \frac{1}{24}\epsilon^{\gamma_1\cdots\gamma_4} BH_{\gamma_1\cdots\gamma_4} \,.
\end{aligned} \tag{5.37}$$

In addition, the Bianchi identities satisfy

$$BF_{+\alpha}{}^{\alpha} = 0 \;, \quad BF_{+\alpha_1\alpha_2} = \frac{1}{2}\epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2} BF_{+\bar\beta_1\bar\beta_2} \tag{5.38}$$

and

$$\begin{aligned}
&BH_{+\alpha_1\alpha_2\alpha_3} = BH_{+\alpha_1\alpha_2\bar\beta} = BH_{-+\gamma}{}^{\gamma} = 0 \;, \quad BH_{-+\alpha_1\alpha_2} = \frac{1}{2}\epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2} BH_{-+\bar\beta_1\bar\beta_2} \;, \\
&BH_{\alpha_1\alpha_2\gamma}{}^{\gamma} = \frac{1}{2}\epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2} BH_{\bar\beta_1\bar\beta_2\gamma}{}^{\gamma} \;, \quad \epsilon^{\bar\gamma_1\cdots\bar\gamma_4} BH_{\bar\gamma_1\cdots\bar\gamma_4} - \epsilon^{\gamma_1\cdots\gamma_4} BH_{\gamma_1\cdots\gamma_4} = 0 \,.
\end{aligned} \tag{5.39}$$

It is significant to see that the Bianchi identities are restricted. This effects the consistency of the theory when the heterotic anomaly and the higher order corrections are considered.

However, if one works at the lowest order, one can impose the Bianchi identities, $BH = BF = 0$. In such a case, all field equations are implied provided that in addition one imposes[9] $E_{--} = 0$, $LH_{-A} = 0$ and $LF_{-} = 0$.

6. $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$ invariant spinors

6.1 Supersymmetry conditions

We have shown in section 3 that the $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors can be written as

$$\begin{aligned} \epsilon_1 &= f(1 + e_{1234}) \ , \\ \epsilon_2 &= g_1(1 + e_{1234}) + ig_2(1 - e_{1234}) \, . \end{aligned} \tag{6.1}$$

The Killing spinors equations for the first spinor have been investigated in the previous section. The gravitino Killing spinor equation for the second spinor can be written as

$$g_2^{-1} g_1 \partial_A \log(g_1 f^{-1})(1 + e_{1234}) + i\partial_A \log g_2(1 - e_{1234}) + i\hat{\nabla}_A(1 - e_{1234}) = 0 \, . \tag{6.2}$$

This equation can be expanded in the basis given in (A.9). In particular, the components along the 1 and e_{1234} directions are

$$\begin{aligned} g_2^{-1} g_1 \partial_A \log(g_1 f^{-1}) + i\partial_A \log g_2 + \frac{i}{2}\hat{\Omega}_{A,\alpha}{}^{\alpha} + \frac{i}{2}\hat{\Omega}_{A,-+} &= 0 \ , \\ g_2^{-1} g_1 \partial_A \log(g_1 f^{-1}) - i\partial_A \log g_2 + \frac{i}{2}\hat{\Omega}_{A,\alpha}{}^{\alpha} - \frac{i}{2}\hat{\Omega}_{A,-+} &= 0 \, . \end{aligned} \tag{6.3}$$

These in turn imply that

$$\begin{aligned} \partial_A \log(g_1 f^{-1}) &= 0 \ , \\ \partial_A \log(g_2 f^{-1}) &= 0 \, . \end{aligned} \tag{6.4}$$

To derive the latter, we have also used the equation that we have obtained for f in the $N = 1$ case. Since the Killing spinors are specified up to a constant scale, they can be written as

$$\begin{aligned} \epsilon_1 &= f(1 + e_{1234}) \\ \epsilon_2 &= f[\cos\varphi(1 + e_{1234}) + i\sin\varphi(1 - e_{1234})] \ , \end{aligned} \tag{6.5}$$

where φ is a constant angle. The spinors ϵ_1 and ϵ_2 must be linearly independent and so the angle φ should satisfy $\sin\varphi \neq 0$. The remaining conditions for the second Killing spinor are as those we have derived for the $N = 1$ case with the difference that the terms proportional to the Levi-Civita tensor epsilon have an additional relative minus sign. Combining, the conditions we have derived for the ϵ_1 Killing spinors with those of the ϵ_2 Killing spinors, we find that the independent conditions associated with the gravitino Killing spinor equation

[9] The set of field equations that should be imposed in addition to the Killing spinor equations is not uniquely defined.

are

$$\partial_A \log f + \frac{1}{2}\hat{\Omega}_{A,-+} = 0 \ , \tag{6.6}$$

$$\hat{\Omega}_{A,\bar{\alpha}\bar{\beta}} = \hat{\Omega}_{A,\alpha}{}^{\alpha} = 0 \ , \tag{6.7}$$

$$\hat{\Omega}_{A,+\bar{\alpha}} = \hat{\Omega}_{A,+\alpha} = 0\,. \tag{6.8}$$

It remains to find the conditions that arise from the dilatino Killing spinor equation. We have already computed the dilatino Killing spinor equation on the spinor ϵ_1 in the previous section. So it remains to find the conditions for ϵ_2. It is straightforward to observe using the results we have derived for the dilatino Killing spinor equation of ϵ_1 that it suffices to compute the dilatino Killing spinor equation on $1 - e_{1234}$. In turn the conditions that arise can be easily read from those on ϵ_1. The only difference is a relative minus sign for the terms proportional to the Levi-Civita tensor epsilon. Combining the conditions for the dilatino Killing spinor equation for both ϵ_1 and ϵ_2 spinors, we find that

$$\partial_{\bar{\alpha}}\Phi - \frac{1}{2}H_{\bar{\alpha}\beta}{}^{\beta} - \frac{1}{2}H_{-+\bar{\alpha}} = 0 \ , \tag{6.9}$$

$$H_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} = 0 \ , \tag{6.10}$$

$$\partial_+\Phi = 0 \ , \tag{6.11}$$

$$H_{+\alpha}{}^{\alpha} = 0 \ , \qquad H_{+\bar{\alpha}_1\bar{\alpha}_2} = 0\,. \tag{6.12}$$

This concludes the analysis of the Killing spinor equations.

6.2 Geometry

6.2.1 Holonomy of $\hat{\nabla}$ connection and supersymmetry

Applying the general arguments presented in 4.1 to this case, one expects that the gravitino Killing spinor equation implies that the holonomy of the $\hat{\nabla}$ connection is contained in $SU(4) \ltimes \mathbb{R}^8$. This can be explicitly seen in the gauge $f = 1$. This gauge can be attained by using the $Spin(9,1)$ gauge transformation $e^{b\Gamma^{05}}$ for $b = \log|f|$ as in the $Spin(7) \ltimes \mathbb{R}^8$ case that we have already investigated. In the gauge $f = 1$, one has

$$\hat{\Omega}_{A,+-} = 0 \ , \tag{6.13}$$

which together with (6.8) imply that all the components of $\hat{\Omega}_{A,+B} = 0$. It is then easy to see that the remaining components of the connection one-form, $\hat{\Omega} = \hat{\Omega}_A e^A$, take values in $su(4) \oplus_s \mathbb{R}^8$. In the presence of fluxes, the Levi-Civita connection of these backgrounds does not have $SU(4) \ltimes \mathbb{R}^8$ holonomy.

We have seen above that we can choose $f = 1$. In addition, the angle that the Killing spinors (6.5) depend on can be eliminated with a constant $GL(2,\mathbb{R})$ transformation. So the Killing spinors can be written as $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = i(1 - e_{1234})$. This is in agreement with the general arguments we have presented in section 4.1 that in the heterotic supergravity the Killing spinors can always be chosen to be constant.

A converse statement is also valid. If $\mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$, there are spinors ϵ_1, ϵ_2 which are parallel with respect to $\hat{\nabla}$ and so they satisfy the gravitino Killing spinor equation. Thus the existence of a solution for the gravitino Killing spinor equation can be entirely characterized by the holonomy of $\hat{\nabla}$.

To investigate further the geometry of spacetime, it is convenient to introduce the $\hat{\nabla}$-parallel forms associated with the parallel spinor bilinears. It turns out that most of the fluxes and geometry can be expressed in terms of these bilinears.

6.2.2 Geometry and spacetime forms bilinears

The $\hat{\nabla}$-parallel forms associated with the spinor pair (ϵ_1, ϵ_1) have already been computed and can be found in the previous section. To compute the forms associated with the spinor pairs (ϵ_2, ϵ_2) and (ϵ_1, ϵ_2), we write the metric as in (5.13), i.e.

$$ds^2 = 2\mathrm{e}^+\mathrm{e}^- + 2\delta_{\alpha\bar{\beta}}\mathrm{e}^\alpha\mathrm{e}^{\bar{\beta}} . \tag{6.14}$$

Then, after a normalization of the spinors (6.5) with $1/\sqrt{2}$, we find the one-forms[10]

$$\begin{aligned} \kappa(\epsilon_2, \epsilon_2) &= -f^2\mathrm{e}^- \\ \kappa(\epsilon_1, \epsilon_2) &= -f^2 \cos\varphi\, \mathrm{e}^- \ , \end{aligned} \tag{6.15}$$

a three-form

$$\xi(\epsilon_1, \epsilon_2) = -f^2 \sin\varphi\, \mathrm{e}^- \wedge \omega \ , \tag{6.16}$$

and two five-forms

$$\begin{aligned} \tau(\epsilon_2, \epsilon_2) &= -f^2\, \mathrm{e}^- \wedge [\mathrm{Re}\,(e^{2i\varphi}\chi) - \frac{1}{2}\omega \wedge \omega] \\ \tau(\epsilon_1, \epsilon_2) &= -f^2\, \mathrm{e}^- \wedge \mathrm{Re}\,[e^{i\varphi}(\chi - \frac{1}{2}\omega \wedge \omega)] \ , \end{aligned} \tag{6.17}$$

where $\omega = -i\delta_{\alpha\bar{\beta}}\mathrm{e}^\alpha \wedge \mathrm{e}^{\bar{\beta}}$ and $\chi = 4\,\mathrm{e}^1 \wedge \mathrm{e}^2 \wedge \mathrm{e}^3 \wedge \mathrm{e}^4$. The above forms can be simplified in the gauge $f = 1$, $\cos\varphi = 0$, $\sin\varphi = 1$. It can be easily seen that if ϵ_1 and ϵ_2 are linearly independent, i.e. $\sin\varphi \neq 0$, the ring of spacetime form bilinears is generated by $\kappa = f^2\mathrm{e}^-$, $\xi = \kappa \wedge \omega$, $\tau_1 = \kappa \wedge \omega \wedge \omega$ and $\tau_2 = \kappa \wedge \chi$. This ring is nilpotent as in the $Spin(7) \ltimes \mathbb{R}^8$ case and

$$\hat{\nabla}\kappa = \hat{\nabla}\xi = \hat{\nabla}\tau_1 = \hat{\nabla}\tau_2 = 0 \ , \tag{6.18}$$

i.e. κ, ξ, τ_1 and τ_2 are parallel with respect to the connection $\hat{\nabla}$. As we have already mentioned, the condition $\hat{\nabla}\kappa = 0$ implies that the one-form κ is associated to a *null Killing vector field* X and $d\kappa - i_X H = 0$. The condition (6.12) which arises from the dilatino Killing spinor equation gives that the two-form $i_X H$ takes values in $\mathfrak{su}(4) \oplus_s \mathbb{R}^8$. This in turn implies that X preserves the $SU(4) \ltimes \mathbb{R}^8$ structure, i.e.

$$\mathcal{L}_X\xi = 0 \ , \quad \mathcal{L}_X\tau_1 = 0 \ , \quad \mathcal{L}_X\tau_2 = 0 \, . \tag{6.19}$$

In addition (6.11) implies that the dilaton is invariant under the diffeomorphisms generated by X.

[10] We have made an additional normalization of the spinor bilinears with a factor of $\sqrt{2}$.

6.2.3 Solution of the Killing spinor equations

The conditions arising from the parallel transport equation imply that $\hat\nabla$ has holonomy contained in $SU(4)\ltimes\mathbb{R}^8$. The decomposition of the fluxes in $SU(4)\ltimes\mathbb{R}^8$ representations is manifest in this case. Nevertheless observe that under $SU(4)$ the space of two-forms decomposes as $\Lambda^2(\mathbb{R}^8)\otimes\mathbb{C}=\Lambda^{2,0}_{\mathbf{6}}\oplus\Lambda^{0,2}_{\mathbf{6}}\oplus\Lambda^{1,1}_{\mathbf{1}}\oplus\Lambda^{1,1}_{\mathbf{15}}$ and the space of three-forms decomposes as $\Lambda^3(\mathbb{R}^8)\otimes\mathbb{C}=\Lambda^{3,0}_{\mathbf{4}}\oplus\Lambda^{0,3}_{\mathbf{4}}\oplus\Lambda^{2,1}_{\mathbf{20}}\oplus\Lambda^{1,2}_{\mathbf{20}}\oplus\Lambda^{2,1}_{\mathbf{4}}\oplus\Lambda^{1,2}_{\mathbf{4}}$. Using these, in the gauge $f=1$, the conditions that arise from gravitino Killing spinor equations can be written as

$$\hat\Omega_{A,+B}=0\;,\quad \hat\Omega^{2,0}_{A,ij}=\hat\Omega^{0,2}_{A,ij}=0\;,\quad \hat\Omega^{\mathbf{1}}_{A,ij}=0 \tag{6.20}$$

and, similarly, the conditions that arise from the dilatino Killing spinor equations can be written as

$$\begin{aligned}&\partial_+\Phi=0\;,\quad 2\partial_i\Phi-\theta_i-H_{-+i}=0\;,\quad H^{3,0}_{ijk}=H^{0,3}_{ijk}=0\;,\\ &H^{\mathbf{1}}_{+ij}=0\;,\qquad H^{2,0}_{+ij}=H^{0,2}_{+ij}=0\;,\end{aligned} \tag{6.21}$$

where the restriction to representations of $SU(4)$ is referred to the i,j,k indices, θ is the Lee form

$$\theta_i=-\nabla^k\omega_{kj}I^j{}_i\;, \tag{6.22}$$

and the endomorphism I is defined by $\omega_{ij}=g_{ik}I^k{}_j$. To rewrite the conditions that arise from the dilatino Killing spinor equation in terms of the Lee form, we have used $\hat\nabla_i\omega_{jk}=0$.

The conditions (6.20) and (6.21) can be solved to express the fluxes in terms of the geometry. As we have mentioned already, the first condition in (6.20) implies that $i_XH=d\kappa$. The remaining conditions imply that $\hat\nabla_A\omega_{ij}=0$ and $\hat\nabla_A\chi_{ijkl}=0$. The former condition implies that

$$\hat\nabla_-\omega_{ij}=0\;,\quad \hat\nabla_i\omega_{jk}=0\,. \tag{6.23}$$

These two equations can be solved using, $H^{3,0}_{ijk}=H^{0,3}_{ijk}=0$, to give

$$\begin{aligned}&H_{-ij}-H_{-kl}I^k{}_iI^l{}_j=-2I^m{}_i\nabla_-\omega_{mj}\;,\\ &H_{ijk}=-3I^m{}_{[i}(\nabla_j\omega_{k]m}+\nabla_{|m|}\omega_{jk]}-\nabla_k\omega_{j]m})\,.\end{aligned} \tag{6.24}$$

In addition $\hat\nabla_-\chi=0$ gives

$$H_{-\alpha}{}^{\alpha}=\frac{1}{8\cdot 4!}\bar\chi^{ijkl}\nabla_-\chi_{ijkl}\,. \tag{6.25}$$

Therefore all the fluxes apart from $H^{\mathbf{15}}_{-ij}$ are determined in terms of the geometry and the form Killing spinor bilinears. Of course the remaining conditions impose additional restrictions on the metric and torsion. In particular, the component of H in $\Lambda^{2,1}_{\mathbf{4}}$ is related to the Lee form θ and so to the derivative of the dilaton.

Therefore, the metric and torsion can be written as

$$\begin{aligned}ds^2&=2\mathrm{e}^-\mathrm{e}^++\delta_{ij}e^ie^j\;,\\ H&=\mathrm{e}^+\wedge d\kappa-\frac12 I^m{}_i\nabla_-\omega_{mj}\,\mathrm{e}^-\wedge e^i\wedge e^j-\frac{1}{64\cdot 4!}\mathrm{Im}(\bar\chi^{klmn}\nabla_-\chi_{klmn})\,\omega_{ij}\,\mathrm{e}^-\wedge e^i\wedge e^j\\ &\quad+\frac12 H^{\mathbf{15}}_{-ij}\,\mathrm{e}^-\wedge e^i\wedge e^i+\frac{1}{3!}H_{ijk}e^i\wedge e^j\wedge e^k\;,\end{aligned} \tag{6.26}$$

where H_{ijk} is given in the second equation of (6.24) and $d\kappa$ takes values in $\mathfrak{su}(4)\oplus_s\mathbb{R}^8$.

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6.2.4 Local coordinates, distributions and a deformation family

Using similar arguments to those we have presented for the $Spin(7) \ltimes \mathbb{R}^8$ case and introducing coordinates along the null Killing vector $X = \frac{\partial}{\partial u}$, we can write the spacetime metric in the gauge $f = 1$ as

$$ds^2 = 2(dv + m_I dy^I)(du + V dv + n_I dy^I) + \gamma_{IJ} dy^I dy^J \ , \tag{6.27}$$

where all the components are functions of v, y^I and $i_X H = d\mathrm{e}^- = dm$. In addition the second equation in (6.12) implies that dm takes values in $\mathfrak{su}(4) \oplus_s \mathbb{R}^8$. The coordinate v can also be introduced as in the $Spin(7) \ltimes \mathbb{R}^8$ case. One can adapt a frame to the above metric as in (5.23).

Another aspect of the geometry of the spacetime is that it admits two integrable distributions of codimension five. These are spanned by the the one forms $(\mathrm{e}^-, \mathrm{e}^\alpha)$ and $(\mathrm{e}^-, \mathrm{e}^{\bar{\alpha}})$. This can be seen by using the conditions that arise from the gravitino and dilatino Killing spinor equations. This implies that the spacetime admits a "Lorentzian" holomorphic structure. In fact, most of the conditions that arise from the dilatino Killing spinor equation are implied by the integrability of these distributions.

As in the case of a $Spin(7) \ltimes \mathbb{R}^8$-invariant spinor, the spacetime can be thought of as a two parameter deformation family of an eight-dimensional manifold B. The metric (6.27) can be written as the metric on the family by introducing a non-linear connection A whose components are related to m and n as in (5.30). It remains to investigate the geometry of B. We adapt a frame E^A to the metric of the family as in (5.31) and define the spacetime metric $d\tilde{s}^2 = ds^2|_B$, $\tilde{H} = H|_B$, $\tilde{\omega} = \omega|_B$ and $\tilde{\chi} = \chi|_B$ on B. However, as in the $Spin(7)$ case, the forms $\tilde{\omega}$ and $\tilde{\chi}$ are not always parallel with respect to the connection $\hat{\tilde{\nabla}}$ of B with torsion $\tilde{H}$. Therefore although B has an $SU(4)$-structure, it is not compatible with the connection $\hat{\tilde{\nabla}}$.

There is a special case where the $SU(4)$-structure of B is compatible with the connection $\hat{\tilde{\nabla}}$. This appears whenever the rotation of the the null parallel vector field X vanishes, i.e. when $d\kappa = d\mathrm{e}^- = 0$. Using the relation (5.33) between the frames $(\mathrm{e}^-, \mathrm{e}^+, e^i)$ and (E^-, E^+, E^i), and arguments similar to those of the $Spin(7)$ case, one can show that B is a conformally balanced KT manifold equipped with a compatible $SU(4)$ structure, i.e. B is complex,[11] $\tilde{\omega}$ and $\tilde{\chi}$ define an $SU(4)$-structure and they are parallel with respect to $\hat{\tilde{\nabla}}$, i.e. $\hat{\tilde{\nabla}}\tilde{\omega} = 0$ and $\hat{\tilde{\nabla}}\tilde{\chi} = 0$, and that

$$\tilde{\theta} = 2d\tilde{\Phi} \ , \tag{6.28}$$

where $\tilde{\theta} = - \star (\star d\tilde{\omega} \wedge \tilde{\omega})$ is the Lee form of B and $\star$ is the Hodge duality operator on B associated with the volume form $d\mathrm{vol}(B) = \tilde{e}^1 \wedge \ldots \wedge \tilde{e}^4 \wedge \tilde{e}^6 \wedge \ldots \wedge \tilde{e}^9$. Note that $\tilde{e}^i = e^i|_B = E^i|_B$. One can show that B is a complex submanifold of the Lorentzian holomorphic manifold M by using the integrable distributions $(\mathrm{e}^-, \mathrm{e}^\alpha)$ and $(\mathrm{e}^-, \mathrm{e}^{\bar{\alpha}})$ mentioned

[11] Note that there are two-dimensional sigma models with extended world-volume supersymmetry and target spaces which are almost complex manifolds [30].

above. It turns out that all $2n$-dimensional manifolds with an $SU(n)$-structure and skew-symmetric Nijenhuis tensor admit a compatible connection with skew-symmetric torsion. In particular, the torsion of the eight-dimensional manifold B is given as

$$\tilde{H} = -i_{\tilde{I}} d\tilde{\omega} = \star(d\tilde{\omega} \wedge \tilde{\omega}) - \frac{1}{2} \star (\tilde{\theta} \wedge \tilde{\omega} \wedge \tilde{\omega}) \, . \tag{6.29}$$

Examples of such manifolds have been given in [5, 15, 17, 16]. Of course for applications to the heterotic string one has to impose the (generalized) Bianchi identity for H.

The geometry of B can also be described using G-structures. The $SU(4)$-structures on an eight-dimensional manifold can be found by decomposing $\tilde{\nabla}\tilde{\omega}$ and $\tilde{\nabla}\tilde{\chi}$ in irreducible $SU(4)$ representations. In the decomposition of $\tilde{\nabla}\tilde{\omega}$ and $\tilde{\nabla}\tilde{\chi}$ five irreducible $SU(4)$ representations appear labelled by $W_1, \ldots, W_5$, so there are 2^5 $SU(4)$-structures. One can also recover these representations in the decomposition of $d\tilde{\omega}$ and $d\tilde{\chi}$. In particular, one can show that $d\tilde{\omega}^{3,0}$ and $d\chi^{3,2}$ determine $\tilde{\nabla}_\alpha \tilde{\omega}_{\beta\gamma}$ and correspond to the W_1 and W_2 classes respectively. The traceless part of $d\tilde{\omega}^{2,1}$ is associated with the W_3 class and determines the traceless part of of $\nabla_{\bar{\alpha}} \tilde{\omega}_{\beta\gamma}$. Furthermore the trace part of $d\tilde{\omega}^{2,1}$ and the trace part of $d\chi^{4,1}$ determine the trace parts of $\tilde{\nabla}_{\bar{\alpha}} \tilde{\omega}_{\beta\gamma}$ and $\tilde{\nabla}_{\bar{\alpha}} \tilde{\chi}_{\beta_1 \ldots \beta_4}$, respectively, and are associated with the W_4 and W_5 classes. The classes W_4 and W_5 are characterized by the Lee forms $\tilde{\theta}_{\tilde{\omega}}$ and $\tilde{\theta}_{\mathrm{Re}\tilde{\chi}}$ of $\tilde{\omega}$ and $\mathrm{Re}(\tilde{\chi})$, respectively. The Lee form $\tilde{\theta}_{\tilde{\omega}}$ has been given below (6.28), $\tilde{\theta}_{\tilde{\omega}} = \tilde{\theta}$, and the Lee form of $\mathrm{Re}\tilde{\chi}$ is defined as $\tilde{\theta}_{\mathrm{Re}\tilde{\chi}} = -\frac{1}{4} \star (\star d\mathrm{Re}\tilde{\chi} \wedge \mathrm{Re}\tilde{\chi})$. The remaining components of $\tilde{\nabla}\tilde{\omega}$ and $\tilde{\nabla}\tilde{\chi}$ vanish. The above is a generalization of the results of [43] for the $SU(3)$ case, see also [44]. The further generalization to all $SU(n)$-structures is straightforward. Returning to the geometry of the deformed manifold B, since B is complex, $W_1 = W_2 = 0$. In addition, one can show that

$$\tilde{\theta}_{\tilde{\omega}} = \tilde{\theta}_{\mathrm{Re}\tilde{\chi}} = 2d\tilde{\Phi} \, . \tag{6.30}$$

This condition is reminiscent to a condition found in [11] in the context of $\mathbb{R}^{3,1} \times X_6$ heterotic string backgrounds, where X_6 has an $SU(3)$-structure.

6.3 Field equations

As was explained for the $N = 1$ case, it is straightforward to derive the field equations that are implied from the integrability conditions of the Killing spinor equations. In particular, we find for the case with $SU(4) \ltimes \mathbb{R}^8$ invariant spinors that

$$\begin{aligned}
E_{++} &= E_{+\alpha} = E_{\alpha\beta} = 0 \ , \\
E_{-+} &= -\frac{1}{2} e^{2\Phi} L H_{-+} \ , \\
E_{-\alpha} &= -\frac{1}{2} e^{2\Phi} L H_{-\alpha} + \frac{1}{2} B H_{-\alpha\gamma}{}^{\gamma} \ , \\
E_{\alpha\bar{\beta}} &= -\frac{1}{2} B H_{\alpha\bar{\beta}\gamma}{}^{\gamma} \ , \\
L H_{+\alpha} &= L H_{\alpha_1\alpha_2} = 0 \ , \\
e^{2\Phi} L H_{\alpha\bar{\beta}} &= -B H_{-+\alpha\bar{\beta}} \ , \\
L F_{+} &= 0 \ , \\
e^{2\Phi} L F_{\alpha} &= -B F_{-+\alpha} + B F_{\alpha\gamma}{}^{\gamma} \ ,
\end{aligned}$$

$$L\Phi = \frac{1}{2}e^{2\Phi}LH_{-+} + \frac{1}{4}BH_{\gamma}{}^{\gamma}{}_{\delta}{}^{\delta}\,. \tag{6.31}$$

In addition, the Bianchi identities satisfy

$$BF_{+\alpha}{}^{\alpha} = BF_{+\alpha_1\alpha_2} = BF_{\alpha_1\alpha_2\alpha_3} = 0\,, \tag{6.32}$$

and

$$\begin{aligned} BH_{-\alpha_1\alpha_2\alpha_3} &= BH_{+\alpha_1\alpha_2\alpha_3} = BH_{+\alpha_1\alpha_2\bar{\beta}} = BH_{-+\alpha_1\alpha_2} = BH_{-+\gamma}{}^{\gamma} = 0\,, \\ BH_{\alpha_1\alpha_2\alpha_3\alpha_4} &= BH_{\alpha_1\alpha_2\alpha_3\bar{\beta}} = BH_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2} = 0\,. \end{aligned} \tag{6.33}$$

As can be seen from the conditions above, if we choose to impose the Bianchi identities $BF = BH = 0$, the only field equations that remain to be solved are $E_{--} = 0$, $LH_{-A} = 0$ and $LF_{-} = 0$.

7. $N = 2$ with G_2 invariant spinors

7.1 Supersymmetry conditions

The two Killing spinors can be chosen as, see section 3,

$$\epsilon_1 = f(1 + e_{1234})\,, \quad \epsilon_2 = g(e_{15} + e_{2345})\,. \tag{7.1}$$

We have already derived the conditions required for ϵ_1 to be a Killing spinor when we investigated the backgrounds with one supersymmetry. Therefore it remains to derive the conditions for ϵ_2 to be a Killing spinor. After some computation, the gravitino Killing spinor equation gives

$$\hat{\Omega}_{A,-1} = 0\,, \qquad \hat{\Omega}_{A,-\bar{n}} = 0\,, \tag{7.2}$$

$$\partial_A \log g - \frac{1}{2}\hat{\Omega}_{A,1\bar{1}} + \frac{1}{2}\hat{\Omega}_{A,n}{}^{n} - \frac{1}{2}\hat{\Omega}_{A,-+} = 0\,, \tag{7.3}$$

$$\hat{\Omega}_{A,\bar{n}1} - \frac{1}{2}\hat{\Omega}_{A,pm}\epsilon^{pm}{}_{\bar{n}} = 0\,, \tag{7.4}$$

$$\partial_A \log g + \frac{1}{2}\hat{\Omega}_{A,1\bar{1}} - \frac{1}{2}\hat{\Omega}_{A,n}{}^{n} - \frac{1}{2}\hat{\Omega}_{A,-+} = 0\,, \tag{7.5}$$

where $m, n, p, q, \ldots = 2, 3, 4$ and $\epsilon_{mnp} = \epsilon_{1mnp}$. In addition, the dilatino Killing spinor equation gives

$$2\partial_{-}\Phi + H_{-1\bar{1}} - H_{-n}{}^{n} = 0\,, \tag{7.6}$$

$$2\partial_{-}\Phi - H_{-1\bar{1}} + H_{-n}{}^{n} = 0\,, \tag{7.7}$$

$$2H_{-1\bar{m}} + H_{-np}\epsilon^{np}{}_{\bar{m}} = 0\,, \tag{7.8}$$

$$-2\partial_1\Phi + H_{1n}{}^{n} - H_{-+1} - \frac{1}{3}H_{npm}\epsilon^{npm} = 0\,, \tag{7.9}$$

$$\partial_{\bar{n}}\Phi + \frac{1}{2}H_{1\bar{1}\bar{n}} - \frac{1}{2}H_{\bar{n}p}{}^{p} + \frac{1}{2}H_{-+\bar{n}} - \frac{1}{2}H_{pm\bar{1}}\epsilon^{pm}{}_{\bar{n}} = 0\,. \tag{7.10}$$

Comparing the above equations with those derived for the ϵ_1 Killing spinor, we find that the parallel transport equation gives

$$\partial_A \log f + \frac{1}{2}\hat{\Omega}_{A,-+} = 0 \ , \tag{7.11}$$

$$\partial_A \log fg = 0 \ , \tag{7.12}$$

$$\hat{\Omega}_{A,1}{}^{1} = 0 \ , \tag{7.13}$$

$$\hat{\Omega}_{A,n}{}^{n} = 0 \ , \tag{7.14}$$

$$\hat{\Omega}_{A,+\alpha} = \hat{\Omega}_{A,-\alpha} = 0 \ , \quad \alpha,\beta = 1,2,3,4 \ , \tag{7.15}$$

$$\hat{\Omega}_{A,\bar{\alpha}\bar{\beta}} = \frac{1}{2}\hat{\Omega}_{A,\gamma\delta}\epsilon^{\gamma\delta}{}_{\bar{\alpha}\bar{\beta}} \ , \tag{7.16}$$

$$\hat{\Omega}_{A,1\bar{n}} = -\hat{\Omega}_{A,\bar{1}\bar{n}} \ , \tag{7.17}$$

and the dilatino Killing spinor equation gives

$$\partial_+\Phi = \partial_-\Phi = (\partial_1 + \partial_{\bar{1}})\Phi = 0 \ , \tag{7.18}$$

$$\partial_{\bar{1}}\Phi = -\frac{1}{12}H_{\bar{n}\bar{p}\bar{m}}\epsilon^{\bar{n}\bar{p}\bar{m}} + \frac{1}{12}H_{npm}\epsilon^{npm} \ , \tag{7.19}$$

$$\partial_{\bar{n}}\Phi = -\frac{1}{2}H_{\bar{n}}{}^{p}{}_{p} + \frac{1}{4}H_{\bar{1}pm}\epsilon^{pm}{}_{\bar{n}} - \frac{1}{4}H_{1pm}\epsilon^{pm}{}_{\bar{n}} \ , \tag{7.20}$$

$$H_{+\alpha}{}^{\alpha} = 0 \ , \tag{7.21}$$

$$H_{+\bar{\alpha}\bar{\beta}} = \frac{1}{2}H_{+\gamma\delta}\epsilon^{\gamma\delta}{}_{\bar{\alpha}\bar{\beta}} \ , \tag{7.22}$$

$$H_{-1\bar{1}} = H_{-n}{}^{n} \ , \tag{7.23}$$

$$H_{-1\bar{n}} = -\frac{1}{2}H_{-pm}\epsilon^{pm}{}_{\bar{n}} \ , \tag{7.24}$$

$$H_{-+\bar{1}} = H_{\bar{1}}{}^{n}{}_{n} - \frac{1}{6}H_{npm}\epsilon^{npm} - \frac{1}{6}H_{\bar{n}\bar{p}\bar{m}}\epsilon^{\bar{n}\bar{p}\bar{m}} \ , \tag{7.25}$$

$$H_{-+\bar{n}} = \frac{1}{2}H_{1pm}\epsilon^{pm}{}_{\bar{n}} + \frac{1}{2}H_{\bar{1}pm}\epsilon^{pm}{}_{\bar{n}} - H_{\bar{n}1\bar{1}} \ . \tag{7.26}$$

This concludes the analysis of the Killing spinor equations. In the remainder of the section, we shall investigate the geometry of the backgrounds with G_2 invariant spinors.

7.1.1 Holonomy of $\hat{\nabla}$ connection

The gravitino Killing spinor equation implies that the holonomy of the connection $\hat{\nabla}$ is contained in G_2, which is the stability subgroup of the spinors (7.1) in $Spin(9,1)$. One can also see this explicitly. This is easily done in the gauge $f = 1$. As in the previous cases we have already investigated, this gauge can be attained by the $Spin(9,1)$ gauge transformation $e^{b\Gamma^{05}}$ for $b = \log|f|$. Then (7.12) implies that g is also constant and so it can be chosen as $g = 1$. In the gauge $f = g = 1$, one has

$$\hat{\Omega}_{A,+-} = 0 \ , \tag{7.27}$$

which together with (7.15) imply that all the components of $\hat{\Omega}_{A,+B} = 0$. It is then easy to see that (7.12)-(7.17) imply that the remaining components of the connection one-form, $\hat{\Omega} = \hat{\Omega}_A e^A$, take values in $\mathfrak{g}_2$. The Levi-Civita connection does not have G_2 holonomy. The analysis of the geometry of supersymmetric backgrounds with G_2 invariant spinors simplifies in the gauge $f = g = 1$.

Conversely, if the connection $\hat{\nabla}$ has holonomy contained in G_2, then there are spinors $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = (e_{15} + e_{2345})$, up to a $Spin(9,1)$ gauge transformation, which are parallel with respect to $\hat{\nabla}$. Therefore the holonomy of $\hat{\nabla}$ completely characterizes the solution of the gravitino Killing spinor equation.

7.1.2 Spacetime form bilinears

To proceed further in the investigation of the geometry, we compute the spacetime forms associated with the Killing spinor bilinears. The spacetime forms of ϵ_1 have already been described in the previous sections. It remains to compute the forms associated with the spinor pairs (ϵ_2, ϵ_2) and (ϵ_1, ϵ_2). In particular after an additional normalization of the spinors, we find the one-forms

$$\kappa(\epsilon_1, \epsilon_2) = -e^1 \ , \quad \kappa(\epsilon_2, \epsilon_2) = e^0 + e^5 \ , \tag{7.28}$$

the three-form

$$\xi(\epsilon_1, \epsilon_2) = \mathrm{Re}\hat{\chi} + e^6 \wedge \hat{\omega} - e^0 \wedge e^1 \wedge e^5 \ , \tag{7.29}$$

and the five-forms

$$\begin{aligned} \tau(\epsilon_1, \epsilon_2) &= -\mathrm{Re}\hat{\chi} \wedge e^0 \wedge e^5 + \mathrm{Im}\hat{\chi} \wedge e^1 \wedge e^6 + \frac{1}{2} e^1 \wedge \hat{\omega} \wedge \hat{\omega} - \hat{\omega} \wedge e^0 \wedge e^5 \wedge e^6 \\ \tau(\epsilon_2, \epsilon_2) &= -(e^0 + e^5) \wedge [e^1 \wedge \mathrm{Re}\hat{\chi} + e^6 \wedge \mathrm{Im}\hat{\chi} + \frac{1}{2}\hat{\omega} \wedge \hat{\omega} + \hat{\omega} \wedge e^1 \wedge e^6] \ , \end{aligned} \tag{7.30}$$

in the gauge $f = g = 1$. Unlike the previous cases we have investigated, the ring of invariant forms is not nilpotent. It is generated by the one-forms $\kappa = \mathrm{e}^-, \kappa' = \mathrm{e}^+$ and $\hat{\kappa} = e^1 = \frac{1}{\sqrt{2}}(\mathrm{e}^1 + \mathrm{e}^{\bar{1}})$, the G_2 invariant form

$$\varphi = \mathrm{Re}\hat{\chi} + e^6 \wedge \hat{\omega} \ , \tag{7.31}$$

and its dual $\star\varphi$, where the Hodge operator is taken with respect to the volume form $e^2 \wedge e^3 \wedge e^4 \wedge e^6 \wedge e^7 \wedge e^8 \wedge e^9$.

The one-forms $\kappa = \mathrm{e}^-, \kappa' = \mathrm{e}^+$ and $\hat{\kappa} = e^1 = \frac{1}{\sqrt{2}}(\mathrm{e}^1 + \mathrm{e}^{\bar{1}})$, in the gauge $f = g = 1$, are associated with the Killing vector fields $X = \mathrm{e}_+, Y = \mathrm{e}_-$ and $Z = e_1$, respectively. This follows from the conditions $\hat{\Omega}_{A,+B} = \hat{\Omega}_{A,-B} = \hat{\Omega}_{A,1B} + \hat{\Omega}_{A,\bar{1}B} = 0$ which summarize (7.11), (7.15) and (7.17), in the gauge $f = g = 1$, and the skew-symmetry of the torsion H. The commutators of these Killing vector fields are

$$\begin{aligned} [X, Y] &= -H^A{}_{+-} e_A \\ [X, Z] &= -H^A{}_{+1} e_A \end{aligned}$$

$$[Y,Z] = -H^{A}{}_{-1}e_A\,. \tag{7.32}$$

The components of the torsion which appear in (7.32) are not required to vanish by the Killing spinor equations. So the above commutators do not vanish and therefore the Killing vector fields do not necessarily commute. As we have shown in 4.2, if two vector fields X, Y are $\hat{\nabla}$-parallel their commutator $[X, Y]$ is $\hat{\nabla}$-parallel as well. So if the commutators of X, Y, Z are independent vector fields, then the spacetime admits up to six parallel vector fields not counting the further commutators that one can construct. So there is a large class of geometries that can occur ranging from a spacetime with three commuting Killing vector fields X, Y, Z to a spacetime that is a Lorentzian Lie group of dimension ten equipped with a left-invariant metric g and a left-invariant closed three form H. We shall not attempt to investigate the full range of possibilities. Instead, we shall focus on the case for which the vector fields X, Y and Z span a Lie algebra under the commutators (7.32).

7.1.3 Backgrounds with three isometries and supersymmetry conditions

Let $\mathfrak{h}$ be the Lie algebra spanned by X, Y and Z. Then $[h, h] \subset h$ implies that

$$H_{-+i} = H_{-1i} = H_{+1i} = 0\ , \quad i, j, k, l = 2, 3, 4, 6, 7, 8, 9\,. \tag{7.33}$$

Therefore the structure constants of the Lie algebra are given by the H_{-+1} component of the torsion. Since H is a three-form, $\mathfrak{h}$ can be either isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, if $H_{-+1} = 0$, or isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, if $H_{-+1} \neq 0$. The analysis can be done for both cases simultaneously.

First consider the consequences of (7.33) on the gravitino Killing spinor equation. It is straightforward to find that (7.11)-(7.17) can be rewritten as

$$\hat{\Omega}_{A,aB} = 0\ , \quad \hat{\Omega}^{\mathbf{7}}_{A,ij} = 0\ , \quad a = -, +, 1\ , \tag{7.34}$$

where we have used the decomposition of $\Lambda^2(\mathbb{R}^7) = \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{14}}$ under G_2,

$$\begin{aligned} \Lambda^2_{\mathbf{7}} &= \{\star(\star\varphi \wedge \alpha)|\, \alpha \in \Lambda^1(\mathbb{R}^7)\} \\ \Lambda^2_{\mathbf{14}} &= \{\alpha \in \Lambda^2(\mathbb{R}^7)|\, \star(\varphi \wedge \alpha) = -\alpha\} \end{aligned} \tag{7.35}$$

and $\Lambda^2_{\mathbf{14}}$ can be identified with the Lie algebra $\mathfrak{g}_2$ of G_2. Similarly, using (7.33), the conditions implied by the dilatino Killing spinor equation can be rewritten as

$$\begin{aligned} \partial_+\Phi = \partial_-\Phi = \partial_1\Phi &= 0\ , \\ i_X H^{\mathbf{7}}_{ij} = i_Y H^{\mathbf{7}}_{ij} = i_Z H^{\mathbf{7}}_{ij} &= 0\ , \\ \partial_i\Phi + \frac{1}{12} H_{jkl} \star\varphi^{jkl}{}_i &= 0\ , \\ H_{-+1} + \frac{1}{6} H_{ijk}\varphi^{ijk} &= 0\,. \end{aligned} \tag{7.36}$$

The second equation in (7.36) implies that $(i_X H)_{ij}$, $(i_Y H)_{ij}$ and $(i_Z H)_{ij}$ take values in $\mathfrak{g}_2$. This together with (7.33) imply that X, Y and Z leave invariant the forms φ and its dual $\star\varphi$, i.e.

$$\mathcal{L}_X\varphi = \mathcal{L}_Y\varphi = \mathcal{L}_Z\varphi = 0\ , \tag{7.37}$$

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and similarly for $\star\varphi$. The last equation in (7.36) implies that the structure constants of $\mathfrak{h}$ can be identified with the singlet of H under the G_2 decomposition $\Lambda^3(\mathbb{R}^7) = \Lambda^3_{\mathbf{1}} \oplus \Lambda^3_{\mathbf{7}} \oplus \Lambda^3_{\mathbf{27}}$, where

$$\begin{aligned} \Lambda^3_{\mathbf{1}} &= \{r\,\varphi |\, r \in \mathbb{R}\}\ , \\ \Lambda^3_{\mathbf{7}} &= \{\star(\varphi \wedge \alpha) |\, \alpha \in \Lambda^1(\mathbb{R}^7)\}\ , \\ \Lambda^3_{\mathbf{27}} &= \{\alpha \in \Lambda^3(\mathbb{R}^7) |\, \alpha \wedge \varphi = 0,\ \alpha \wedge \star\varphi = 0\} = \{s \in S^2(\mathbb{R}^7) |\, \mathrm{tr}(s) = 0\}\,. \end{aligned} \tag{7.38}$$

In addition the seven-dimensional component of H in the above decomposition is identified with the exterior derivative of the dilaton.

7.1.4 The solution of the Killing spinor equations

The space of supersymmetric backgrounds with G_2 invariant spinors is (locally) a principal bundle P equipped with a connection λ. To see this, we again assume that the algebra $\mathfrak{h}$ spanned by the vector fields[12] X, Y and Z closes under Lie brackets and consider a Lie group $\mathcal{H}$ with Lie algebra $\mathfrak{h}$. Then the spacetime $M = P(\mathcal{H}, B, \pi)$, where the base space B is the space of orbits of $\mathcal{H}$ in M and π is the projection of P onto B. It remains to determine the connection λ. This is identified with the components of the frame e along the X, Y and Z directions, i.e.

$$\lambda^a = e^a\,. \tag{7.39}$$

One can immediately see that λ satisfies the requirements of a connection, i.e. $\lambda^a(X_b) = e^a(X_b) = \delta^a{}_b$ where $\{X_b, b = +, -1\} = \{X, Y, Z\}$, and $\mathcal{L}_{X_b}\lambda^a = H^a{}_{bc}\lambda^c$, where H_{abc} are interpreted as the structure constants of $\mathfrak{h}$. The latter is the infinitesimal expression of the requirement that $R^*_g\lambda = Ad_{g^{-1}}\lambda$, $g \in \mathcal{H}$, of a principal bundle connection, see e.g. [45]. Then the tangent bundle decomposes into the vertical and horizonal subspaces, $TM = TP = T^vP \oplus T^hP$, where T^vP is spanned by the vector field X, Y and Z and T^hP is (locally) spanned by the dual vector fields of the e^i components of the frame because $\lambda^a(e_i) = g^{MN} e^a{}_M e_{iN} = 0$.

To determine the Cartan structure equations for this connection, we use the conditions (7.36) to write (7.34) as

$$\begin{aligned} &\Omega_{a,bc} - \frac{1}{2}H_{abc} = 0\ , \quad \Omega_{i,ab} = 0\ , \quad \Omega_{a,bi} = 0\ , \quad \Omega^{\mathbf{7}}_{i,aj} = 0\ , \quad \Omega_{(i,j)a} = 0\ , \\ &\Omega^{\mathbf{7}}_{a,ij} = 0\ , \quad \hat{\Omega}^{\mathbf{7}}_{k,ij} = 0\ , \end{aligned} \tag{7.40}$$

where the restriction to the seven-dimensional representation is referred to the i, j indices. These in turn give rise to the torsion free conditions

$$\begin{aligned} de^a + \Omega_{b,}{}^a{}_c e^b \wedge e^c + \Omega_{i,}{}^a{}_j e^i \wedge e^j &= 0\ , \\ de^i + \Omega_{j,}{}^i{}_k e^j \wedge e^k + \Omega_{a,}{}^i{}_j e^a \wedge e^j + \Omega_{j,}{}^i{}_a e^j \wedge e^a &= 0\,. \end{aligned} \tag{7.41}$$

The first torsion free condition rewritten in terms of H can be interpreted as the Cartan structure equation for the connection λ. In particular, we have

$$d\lambda^a - \frac{1}{2}H^a{}_{bc}\lambda^b \wedge \lambda^c - \frac{1}{2}H^a{}_{ij} e^i \wedge e^j = 0\,. \tag{7.42}$$

[12] These vector fields do not have fixed points because they are $\hat{\nabla}$-parallel and so they cannot vanish.

Since the curvature $\mathcal{F}$ of a principal bundle connection λ is identified with the horizontal part of $d\lambda$, we find that

$$\mathcal{F}^a = \frac{1}{2} H^a{}_{ij} e^i \wedge e^j \,. \tag{7.43}$$

In addition the condition (7.36), part of which can also be written as $H^{\mathbf{7}}_{aij} = 0$, implies that the curvature $\mathcal{F}^a$ is that of a $\mathfrak{g}_2$ type of instanton on B with gauge Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. In terms of these principal bundle data, the metric ds^2 and torsion H of spacetime can be written as

$$\begin{aligned} ds^2 &= \eta_{ab} \lambda^a \lambda^b + \pi^* d\tilde{s}^2 \\ H &= \frac{1}{3} \eta_{ab} \lambda^a \wedge d\lambda^b + \frac{2}{3} \eta_{ab} \lambda^a \wedge \mathcal{F}^b + \pi^* \tilde{H} \,, \end{aligned} \tag{7.44}$$

where $d\tilde{s}^2$ and $\tilde{H}$ is a metric and a three-form on B and horizontally lifted to P with π, respectively.

It remains to determine the geometry of the base space B of the principal bundle. B is equipped with a metric $d\tilde{s}^2 = \delta_{ij} e^i e^j |_B$ and a three-form $\tilde{H}$ which is the horizontal part of H. Note that $d\tilde{H} \neq 0$. In addition it is equipped with a G_2-invariant three-form $\tilde{\varphi}$ such that $\varphi = \pi^* \tilde{\varphi}$. This is because φ is horizontal and $\mathcal{L}_a \varphi = 0$. Furthermore $\hat{\tilde{\nabla}} \tilde{\varphi} = 0$ which follows from $\hat{\nabla} \varphi = 0$. Therefore B is a Riemannian manifold equipped with a metric connection with skew-symmetric torsion and $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$ and thus admits a G_2-structure. It has been shown in [23] that any seven-dimensional manifold with an integrable G_2-structure admits a unique metric connection $\hat{\tilde{\nabla}}$ with torsion a three-form

$$\tilde{H} = -\frac{1}{6} (d\tilde{\varphi}, \star\tilde{\varphi})\, \tilde{\varphi} + \star d\tilde{\varphi} - \star(\tilde{\theta} \wedge \tilde{\varphi}) \tag{7.45}$$

such that $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$, where

$$\tilde{\theta} = -\frac{1}{3} \star (\star d\tilde{\varphi} \wedge \tilde{\varphi}) \tag{7.46}$$

is the Lee-form and $d\mathrm{vol}(B) = e^2 \wedge e^3 \wedge e^4 \wedge e^6 \wedge e^7 \wedge e^8 \wedge e^9$. An integrable G_2-structure satisfies $d \star \tilde{\varphi} = -\tilde{\theta} \wedge \star\tilde{\varphi}$. In addition the third condition in (7.36) can be rewritten as

$$\tilde{\theta} = 2 d\tilde{\Phi} \tag{7.47}$$

and so B is conformally balanced. If $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, then the last condition implies that the singlet $\tilde{H}^{\mathbf{1}}$ of $\tilde{H}$ in the decomposition $\Lambda^3(\mathbb{R}^7) = \Lambda^3_{\mathbf{1}} \oplus \Lambda^3_{\mathbf{7}} \oplus \Lambda^3_{\mathbf{27}}$ vanishes, $\tilde{H}^{\mathbf{1}} = 0$. This implies that $d\tilde{\varphi}$ is orthogonal to $\star\tilde{\varphi}$. These are precisely the manifolds with G_2-structures investigated in the context of supersymmetric backgrounds in [23]. Moreover it can be shown that these G_2-structures are conformally equivalent to cocalibrated G_2-structures of pure W_3 type.[13] However, if $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$, then $\tilde{H}^{\mathbf{1}}$ is identified with the structure constants of $\mathfrak{sl}(2,\mathbb{R})$ and so $\tilde{H}^{\mathbf{1}} \neq 0$.

[13] The covariant derivative $\tilde{\nabla}\tilde{\varphi}$ can be decomposed into four irreducible G_2 representations W_1, W_2, W_3 and W_4 which are determined by $d\tilde{\varphi}$ and $d \star \tilde{\varphi}$ [46], see also [43]. So there are sixteen G_2-structures on a seven-dimensional manifold.

To summarize, the solution of the Killing spinor equations for the backgrounds that we have investigated above can be described as follows: The spacetime is (locally) a principal bundle $P(\mathcal{H}, B, \pi)$. The group $\mathcal{H}$ of the fibre has Lie algebra either $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ or $\mathfrak{sl}(2, \mathbb{R})$, and P is equipped with a connection λ whose curvature $\mathcal{F}$ is a $\mathfrak{g}_2$ instanton. The base space B is a seven-dimensional manifold equipped with a metric connection with skew-symmetric torsion $\hat{\tilde{\nabla}}$ and $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$ and the associated G_2 structure is conformally balanced, i.e. it satisfies the conditions $d \star \tilde{\varphi} = -\tilde{\theta} \wedge \star\tilde{\varphi}$ and $\tilde{\theta} = 2d\tilde{\Phi}$. The metric and torsion are given by (7.44), and the dilaton Φ is a function of B. If in addition $\mathcal{H}$ is abelian, then $(d\tilde{\varphi}, \star\tilde{\varphi}) = 0$.

Using the description of the geometry of spacetime in terms of principal bundle data, we can write the exterior derivative of the torsion

$$dH = \eta_{ab}\, \mathcal{F}^a \wedge \mathcal{F}^b + \pi^*\, d\tilde{H}\,. \tag{7.48}$$

The first term in the right-hand-side of dH can be recognized as a representative of the first Pontrjagin class of the principal bundle P. Therefore the non-horizontal part of H is the form that trivializes the first Pontrjagin class of P on the bundle space. If one requires that $dH = 0$, then the representative of the first Pontrjagin class of P should cancel against the contribution from the base space B. Of course if P is a globally defined principal bundle over B, then the condition $dH = 0$ implies that the first Pontrjagin form is exact and therefore the first Pontrjagin class of the principal bundle should vanish. Observe that it is *not* required that $d\tilde{H} = 0$.

7.2 Field equations

7.2.1 Integrability conditions

We shall demonstrate that if the Bianchi identities of H and F are satisfied, then the Killing spinor equations imply all the field equations. To see this, one can show that the integrability conditions of the Killing spinor equations imply

$$\begin{aligned}
E_{-A} &= E_{+A} = 0\,, \quad E_{n1} = -E_{n\bar{1}} = -\frac{1}{2} BH_{n1m}{}^{m}\,, \\
E_{11} &= -E_{1\bar{1}} = \frac{1}{6}\epsilon^{\bar{n}\bar{p}\bar{m}} BH_{\bar{1}\bar{n}\bar{p}\bar{m}}\,, \quad E_{np} = -\frac{1}{2}\epsilon_{(n}{}^{\bar{m}\bar{q}} BH_{p)1\bar{m}\bar{q}}\,, \\
E_{n\bar{p}} &= -\frac{1}{2} BH_{-+n\bar{p}} - \frac{1}{2} BH_{n\bar{p}m}{}^{m} - \frac{1}{4}\epsilon_{n}{}^{\bar{m}\bar{q}} BH_{\bar{p}\bar{m}\bar{q}1} + \frac{1}{4}\epsilon_{\bar{p}}{}^{mq} BH_{nmq1}\,, \\
LH_{-A} &= LH_{+A} = LH_{1A} = 0\,, \quad e^{2\Phi} LH_{np} = BH_{npm}{}^{m} - \epsilon_{np}{}^{\bar{m}} BH_{\bar{m}\bar{1}q}{}^{q}\,, \\
e^{2\Phi} LH_{n\bar{p}} &= BH_{-+n\bar{p}} - \frac{1}{2}\epsilon_{n}{}^{\bar{m}\bar{q}} BH_{\bar{p}\bar{m}\bar{q}1} - \frac{1}{2}\epsilon_{\bar{p}}{}^{mq} BH_{nmq\bar{1}}\,, \\
LF_{-} &= LF_{+} = 0\,, \quad e^{2\Phi} LF_{n} = BF_{-+n} + BF_{np}{}^{p} - BF_{n1\bar{1}} - \epsilon_{n}{}^{\bar{p}\bar{q}} BF_{1\bar{p}\bar{q}}\,, \\
e^{2\Phi} LF_{1} &= BF_{-+\bar{1}} - BF_{\bar{1}n}{}^{n} - \frac{1}{3}\epsilon^{\bar{n}\bar{m}\bar{p}} BF_{\bar{n}\bar{m}\bar{p}}\,, \\
L\Phi &= \frac{1}{4} BH_{p}{}^{p}{}_{m}{}^{m} + \frac{1}{3}\epsilon^{npm} BH_{1npm}
\end{aligned} \tag{7.49}$$

and

$$BF_{-1\bar{1}} = BF_{-n}{}^{n}\,, \quad BF_{+1\bar{1}} = -BF_{+n}{}^{n}\,,$$

$$
\begin{aligned}
&BF_{-+1} - BF_{-+\bar{1}} - BF_{1n}{}^{n} - BF_{\bar{1}n}{}^{n} = 0 \ , \\
&BF_{-+1} - BF_{1n}{}^{n} + \frac{1}{6}\epsilon^{npm} BF_{npm} + \frac{1}{6}\epsilon^{\bar{n}\bar{p}\bar{m}} BF_{\bar{n}\bar{p}\bar{m}} = 0 \ , \\
&BF_{-n\bar{1}} = \frac{1}{2}\epsilon_{n}{}^{\bar{p}\bar{m}} BF_{-\bar{p}\bar{m}} \ , \quad BF_{+n1} = -\frac{1}{2}\epsilon_{n}{}^{\bar{p}\bar{m}} BF_{+\bar{p}\bar{m}} \ , \\
&BF_{-+n} - BF_{n1\bar{1}} - \frac{1}{2}\epsilon_{n}{}^{\bar{p}\bar{m}} BF_{\bar{p}\bar{m}\bar{1}} - \frac{1}{2}\epsilon_{n}{}^{\bar{p}\bar{m}} BF_{\bar{p}\bar{m}1} = 0 \ , \\
&BH_{-ABC} = BH_{+ABC} = BH_{np1\bar{1}} = BH_{1\bar{1}n}{}^{n} = 0 \ , \\
&\epsilon^{\bar{n}\bar{p}\bar{m}} BH_{\bar{1}\bar{n}\bar{p}\bar{m}} = \epsilon^{npm} BH_{1npm} = -\epsilon^{\bar{n}\bar{p}\bar{m}} BH_{1\bar{n}\bar{p}\bar{m}} \ , \\
&\frac{1}{6}\epsilon^{\bar{p}\bar{m}\bar{q}} BH_{n\bar{p}\bar{m}\bar{q}} = BH_{n1p}{}^{p} = -BH_{n\bar{1}p}{}^{p} \ , \\
&BH_{-+n\bar{p}} + \frac{1}{4}\epsilon_{n}{}^{\bar{m}\bar{q}} BH_{\bar{p}\bar{1}\bar{m}\bar{q}} + \frac{1}{4}\epsilon_{n}{}^{\bar{m}\bar{q}} BH_{\bar{p}1\bar{m}\bar{q}} - \frac{1}{4}\epsilon_{\bar{p}}{}^{mq} BH_{nmq\bar{1}} - \frac{1}{4}\epsilon_{\bar{p}}{}^{mq} BH_{nmq1} = 0 \ , \\
&BH_{-+n\bar{p}} + BH_{n\bar{p}1\bar{1}} - \frac{1}{2}\epsilon_{\bar{p}}{}^{mq} BH_{nmq\bar{1}} - \frac{1}{2}\epsilon_{\bar{p}}{}^{mq} BH_{nmq1} = 0 \, .
\end{aligned}
\tag{7.50}
$$

In the above conditions, we have not imposed the Bianchi identity BH of H. In the order of α' that we are working $BH = dH = 0$ and so there is no contribution from the Bianchi identities. But in the next order up in α', the above integrability conditions are believed to hold but $dH \neq 0$. As a result some of the field equations that are derived in the one-loop sigma model approximation can be expressed in terms of dH. This has been used in [10] to investigate heterotic backgrounds taking into account the two-loop and higher order corrections to the field equations.

If we take $BH = BF = 0$, the integrability conditions above imply all field equations. In the absence of the gauge field A, the only Bianchi identity that has to be imposed is that of H. This has been computed in (7.48). We shall explore this to give examples of some supersymmetric backgrounds.

7.2.2 Examples

As an example, let us consider the case where $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. Then

$$
\lambda^a = dx^a + A^a \tag{7.51}
$$

and so

$$
\begin{aligned}
ds^2 &= \eta_{ab}(dx^a + A^a)(dx^b + A^b) + \delta_{ij} e^i e^j \ , \\
H &= \eta_{ab}(dx^a + A^a) \wedge dA^b + \pi^* \tilde{H} \, .
\end{aligned}
\tag{7.52}
$$

If one requires closure of H and choose $\tilde{H} = -\eta_{ab} A^a \wedge dA^b + H_B$, then

$$
H = \eta_{ab}\, dx^a \wedge dA^b + H_B \ , \tag{7.53}
$$

where H_B is a three-form on B such that $dH_B = 0$. Clearly $dH = 0$. Within a brane interpretation of these solutions, the connection A^0 along the time direction is thought of as rotation while the remaining connections are thought of as wrapping.

A special case of this example is whenever the only non-vanishing rotation and wrapping is a along a null direction. In this case, the Chern-Simons form contribution vanishes.

Thus one can set $\tilde{H} = H_B$. Such kind of solutions have been consider before[14] in [47]. The metric and torsion are

$$\begin{aligned} ds^2 &= 2dv(du + A) + dx^2 + \delta_{ij} e^i e^j \ , \\ H &= 2\, dv \wedge dA + H_B \, . \end{aligned} \tag{7.54}$$

In such a case, the base space B is a conformally balanced Riemannian manifold equipped with a connection $\hat{\tilde{\nabla}}$ with torsion a three-form $\tilde{H}$ such that $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$ and $\tilde{H}$ is closed.

The form of dH in (7.48) raises the possibility of canceling the representative of the first Pontrjagin class of $P = M$ against the representatives first Pontrjagin classes of the tangent bundle of M and that of the gauge bundle. This will solve the generalized Bianchi identity of H, schematically $dH \sim \alpha'(\mathrm{tr}R^2 - \mathrm{tr}F^2)$, where dH is given in (7.48). As we have already mentioned consistency in this case requires that the two loop correction to the field equations should be taken into account. Nevertheless the integrability conditions we have derived are still valid because the gravitino, dilatino and gaugino supersymmetry transformations are not believed to receive corrections to this order but $dH \neq 0$. A systematic investigation of such solutions will be given elsewhere.

8. N=3 backgrounds

8.1 Supersymmetry conditions

We have shown in section 3 that the three Killing spinors can be written as

$$\begin{aligned} \epsilon_1 &= f\,(1 + e_{1234}) \ , \\ \epsilon_2 &= g_1(1 + e_{1234}) + i g_2 (1 - e_{1234}) \ , \\ \epsilon_3 &= h_1(1 + e_{1234}) + i h_2 (1 - e_{1234}) + i h_3 (e_{12} + e_{34}) \ , \end{aligned} \tag{8.1}$$

where $f, g_1, g_2, h_1, h_2, h_3$ are spacetime functions.

Using the results we have derived for backgrounds with two supersymmetries, we can write the gravitino Killing spinor equation of ϵ_3 as

$$\begin{aligned} h_1 \partial_A \log(h_1 f^{-1})(1 + e_{1234}) + i h_2 \partial_A \log(h_2 g_2^{-1})(1 - e_{1234}) \\ + i \partial_A h_3 (e_{12} + e_{34}) + i h_3 \hat{\nabla}_A (e_{12} + e_{34}) = 0 \, . \end{aligned} \tag{8.2}$$

Evaluating this equation along 1 and e_{1234}, we find that

$$\begin{aligned} h_1 \partial_A \log(h_1 f^{-1}) + i h_2 \partial \log(h_2 g_2^{-1}) - i h_3 \hat{\Omega}_{A,12} - i h_3 \hat{\Omega}_{A,34} = 0 \ , \\ h_1 \partial_A \log(h_1 f^{-1}) - i h_2 \partial \log(h_2 g_2^{-1}) + i h_3 \hat{\Omega}_{A,\bar{1}\bar{2}} + i h_3 \hat{\Omega}_{A,\bar{3}\bar{4}} = 0 \ , \end{aligned} \tag{8.3}$$

and using (6.4) and (6.7), we get

$$\partial_A \log(h_1 f^{-1}) = 0 \ ,$$

[14] Our current results correct some of the fractions of supersymmetry that have appeared in [47].

$$\partial_A \log(h_2 f^{-1}) = 0\,. \tag{8.4}$$

The remaining conditions of the gravitino Killing spinor equations on ϵ_3 are

$$\partial_A \log h_3 + \frac{1}{2}\hat{\Omega}_{A,-+} = 0\ , \tag{8.5}$$

$$\hat{\Omega}_{A,1\bar{1}} + \hat{\Omega}_{A,2\bar{2}} - \hat{\Omega}_{A,3\bar{3}} - \hat{\Omega}_{A,4\bar{4}} = 0\ , \tag{8.6}$$

$$\hat{\Omega}_{A,+\alpha} = 0\ , \tag{8.7}$$

$$\hat{\Omega}_{A,4\bar{2}} = -\hat{\Omega}_{A,1\bar{3}}\ , \tag{8.8}$$

$$\hat{\Omega}_{A,3\bar{2}} = \hat{\Omega}_{A,1\bar{4}}\,. \tag{8.9}$$

The dilatino Killing spinor equation for ϵ_3 implies the conditions

$$\begin{aligned}
&\partial_+\Phi = 0\ ,\\
&\partial_{\bar{1}}\Phi = -H_{2\bar{3}\bar{4}} + \frac{1}{2}H_{2\bar{2}\bar{1}} - \frac{1}{2}H_{3\bar{3}\bar{1}} - \frac{1}{2}H_{4\bar{4}\bar{1}} - \frac{1}{2}H_{+-\bar{1}}\ ,\\
&\partial_{\bar{2}}\Phi = H_{1\bar{3}\bar{4}} + \frac{1}{2}H_{1\bar{1}\bar{2}} - \frac{1}{2}H_{3\bar{3}\bar{2}} - \frac{1}{2}H_{4\bar{4}\bar{2}} - \frac{1}{2}H_{+-\bar{2}}\ ,\\
&\partial_{\bar{3}}\Phi = -H_{4\bar{1}\bar{2}} - \frac{1}{2}H_{1\bar{1}\bar{3}} - \frac{1}{2}H_{2\bar{2}\bar{3}} + \frac{1}{2}H_{4\bar{4}\bar{3}} - \frac{1}{2}H_{+-\bar{3}}\ ,\\
&\partial_{\bar{4}}\Phi = H_{3\bar{1}\bar{2}} - \frac{1}{2}H_{1\bar{1}\bar{4}} - \frac{1}{2}H_{2\bar{2}\bar{4}} + \frac{1}{2}H_{3\bar{3}\bar{4}} - \frac{1}{2}H_{+-\bar{4}}\ ,\\
&H_{+\bar{1}1} + H_{+\bar{2}2} - H_{+\bar{3}3} - H_{+\bar{4}4} = 0\ ,\\
&H_{+\bar{3}\bar{4}} = -H_{+\bar{1}\bar{2}}\ ,\\
&H_{+\bar{4}2} = -H_{+\bar{1}3}\ ,\\
&H_{+\bar{3}2} = H_{+\bar{1}4}\,.
\end{aligned} \tag{8.10}$$

Combining the above results with the conditions we have derived for the first two Killing spinors ϵ_1, ϵ_2, in section 6.1, the gravitino Killing spinor equations implies the conditions

$$\partial_A \log f + \frac{1}{2}\hat{\Omega}_{A,-+} = 0\ , \tag{8.11}$$

$$\partial_A \log(g_r f^{-1}) = \partial_A \log(h_p f^{-1}) = 0, \quad r = 1,2 \quad p = 1,2,3\ , \tag{8.12}$$

$$\hat{\Omega}_{A,+\bar{\alpha}} = 0\ , \quad \alpha, \beta = 1,2,3,4\ , \tag{8.13}$$

$$\hat{\Omega}_{A,\bar{\alpha}\bar{\beta}} = 0\ , \tag{8.14}$$

$$\hat{\Omega}_{A,1\bar{1}} + \hat{\Omega}_{A,2\bar{2}} = 0\ , \tag{8.15}$$

$$\hat{\Omega}_{A,3\bar{3}} + \hat{\Omega}_{A,4\bar{4}} = 0\ , \tag{8.16}$$

$$\hat{\Omega}_{A,4\bar{2}} = -\hat{\Omega}_{A,1\bar{3}}\ , \tag{8.17}$$

$$\hat{\Omega}_{A,3\bar{2}} = \hat{\Omega}_{A,1\bar{4}}\ , \tag{8.18}$$

and the dilatino Killing spinor gives

$$\partial_+\Phi = 0\ , \tag{8.19}$$

$$\partial_{\bar{1}}\Phi = -\frac{1}{2}H_{2\bar{3}\bar{4}} + \frac{1}{2}H_{2\bar{2}\bar{1}} - \frac{1}{2}H_{+-\bar{1}} \ , \tag{8.20}$$

$$\partial_{\bar{2}}\Phi = \frac{1}{2}H_{1\bar{3}\bar{4}} + \frac{1}{2}H_{1\bar{1}\bar{2}} - \frac{1}{2}H_{+-\bar{2}} \ , \tag{8.21}$$

$$\partial_{\bar{3}}\Phi = -\frac{1}{2}H_{4\bar{1}\bar{2}} + \frac{1}{2}H_{4\bar{4}\bar{3}} - \frac{1}{2}H_{+-\bar{3}} \ , \tag{8.22}$$

$$\partial_{\bar{4}}\Phi = \frac{1}{2}H_{3\bar{1}\bar{2}} + \frac{1}{2}H_{3\bar{3}\bar{4}} - \frac{1}{2}H_{+-\bar{4}} \ , \tag{8.23}$$

$$H_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = 0 \ , \tag{8.24}$$

$$H_{2\bar{3}\bar{4}} + H_{3\bar{3}\bar{1}} + H_{4\bar{4}\bar{1}} = 0 \ , \tag{8.25}$$

$$-H_{1\bar{3}\bar{4}} + H_{3\bar{3}\bar{2}} + H_{4\bar{4}\bar{2}} = 0 \ , \tag{8.26}$$

$$H_{4\bar{1}\bar{2}} + H_{1\bar{1}\bar{3}} + H_{2\bar{2}\bar{3}} = 0 \ , \tag{8.27}$$

$$-H_{3\bar{1}\bar{2}} + H_{1\bar{1}\bar{4}} + H_{2\bar{2}\bar{4}} = 0 \ , \tag{8.28}$$

$$H_{+\bar{\alpha}\bar{\beta}} = 0 \ , \tag{8.29}$$

$$H_{+1\bar{1}} + H_{+2\bar{2}} = 0 \ , \tag{8.30}$$

$$H_{+3\bar{3}} + H_{+4\bar{4}} = 0 \ , \tag{8.31}$$

$$H_{+\bar{4}2} = -H_{+\bar{1}3} \ , \tag{8.32}$$

$$H_{+\bar{3}2} = H_{+\bar{1}4} \, . \tag{8.33}$$

It remains to investigate the geometric properties of $N = 3$ backgrounds which are implied by the above conditions.

8.2 Geometry

8.2.1 The holonomy of $\hat{\nabla}$ connection

As we have explained in previous cases, the holonomy of $\hat{\nabla}$ is contained in the stability subgroup of the Killing spinors in $Spin(9,1)$, which in this case is $Sp(2) \ltimes \mathbb{R}^8$. This can be seen explicitly in the gauge where the spacetime functions f, g_r and h_p in the Killing spinors are constant. It is clear from the supersymmetry conditions (8.12) that for this it suffices to find a gauge $Spin(9,1)$ transformation to set $f = 1$. As in previous cases, this gauge can be attained with a $Spin(9,1)$ gauge transformation in the direction Γ_{05}. In this gauge, (8.11) implies that $\hat{\Omega}_{A,-+} = 0$ and so with (8.13), we have

$$\hat{\Omega}_{A,+B} = 0 \, . \tag{8.34}$$

Then the remaining conditions of the gravitino Killing spinor equations imply that the connection $\hat{\nabla}$ takes values in $\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$ and so the holonomy of the connection is contained in $Sp(2) \ltimes \mathbb{R}^8$. Moreover, we can set $\epsilon_1 = 1 + e_{1234}$, $\epsilon_2 = i(1 - e_{1234})$ and $\epsilon_3 = i(e_{12} + e_{34})$ using a constant $GL(3,\mathbb{R})$ transformation. This is in agreement with the general arguments we have presented in section 4.1. Conversely, if $\mathrm{hol}(\hat{\nabla}) \subseteq Sp(2) \ltimes \mathbb{R}^8$, then there are three parallel spinors which solve the gravitino Killing spinor equation.

8.2.2 Spacetime forms and the geometry of spacetime

We have shown that in the gauge $f = 1$ the Killing spinors can be chosen as $\epsilon_1 = 1 + e_{1234}$, $\epsilon_2 = i(1 - e_{1234})$ and $\epsilon_3 = i(e_{12} + e_{34})$. The spacetime form bilinears associated with the spinors (ϵ_1, ϵ_1), (ϵ_1, ϵ_2) and (ϵ_2, ϵ_2) have already been computed in previous sections. After an additional normalization of the spinors with $1/\sqrt{2}$, we find the non-vanishing spacetime form bilinears of the pairs (ϵ_1, ϵ_3), (ϵ_2, ϵ_3), (ϵ_3, ϵ_3) are the one-forms

$$\kappa(\epsilon_3, \epsilon_3) = e^0 - e^5 \quad , \tag{8.35}$$

the three-forms

$$\begin{aligned} \xi(\epsilon_1, \epsilon_3) &= (e^0 - e^5) \wedge \omega_K \ , \\ \xi(\epsilon_2, \epsilon_3) &= -(e^0 - e^5) \wedge \omega_J \ , \end{aligned} \tag{8.36}$$

and the five-forms

$$\begin{aligned} \tau(\epsilon_1, \epsilon_3) &= -(e^0 - e^5) \wedge \omega_I \wedge \omega_J \ , \\ \tau(\epsilon_2, \epsilon_3) &= -(e^0 - e^5) \wedge \omega_I \wedge \omega_K \ , \\ \tau(\epsilon_3, \epsilon_3) &= -(e^0 - e^5) \wedge \left[4\,\mathrm{Re}\,(\mathrm{e}^{\bar{1}} \wedge \mathrm{e}^{\bar{2}} \wedge \mathrm{e}^3 \wedge \mathrm{e}^4) + \frac{1}{2} \breve{\omega} \wedge \breve{\omega} \right] , \end{aligned} \tag{8.37}$$

where we have set $\omega_I = \omega$,

$$\begin{aligned} \omega_J &= \mathrm{e}^1 \wedge \mathrm{e}^2 + \mathrm{e}^{\bar{1}} \wedge \mathrm{e}^{\bar{2}} + \mathrm{e}^3 \wedge \mathrm{e}^4 + \mathrm{e}^{\bar{3}} \wedge \mathrm{e}^{\bar{4}} \ , \\ \omega_K &= i(\mathrm{e}^1 \wedge \mathrm{e}^2 - \mathrm{e}^{\bar{1}} \wedge \mathrm{e}^{\bar{2}} + \mathrm{e}^3 \wedge \mathrm{e}^4 - \mathrm{e}^{\bar{3}} \wedge \mathrm{e}^{\bar{4}}) \ , \end{aligned} \tag{8.38}$$

and

$$\breve{\omega} = i(\mathrm{e}^1 \wedge \mathrm{e}^{\bar{1}} + \mathrm{e}^2 \wedge \mathrm{e}^{\bar{2}} - \mathrm{e}^3 \wedge \mathrm{e}^{\bar{3}} - \mathrm{e}^4 \wedge \mathrm{e}^{\bar{4}}) \, . \tag{8.39}$$

The forms ω_I, ω_J and ω_K are the familiar two-forms that appear on manifolds with an $Sp(2)$-structure and I, J and K are the associated endomorphisms. The two-form $\breve{\omega}$ does not have an invariant meaning but it is necessary to write the five-form of the $Spin(7) \ltimes \mathbb{R}^8$-structure associated with the spinor η_3.

As in all the previous null supersymmetric backgrounds, the one-form $\kappa = \mathrm{e}^-$ is associated with a null parallel vector field X. In addition the conditions (8.29)-(8.32) of the dilatino Killing spinor equations imply that $i_X H$ takes values in $\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$. This in turn implies that X leaves invariant the $Sp(2) \ltimes \mathbb{R}^8$-structure of the spacetime, i.e.

$$\mathcal{L}_X \alpha = 0 \ , \tag{8.40}$$

where α are all the form bilinears constructed from the parallel spinors.

8.2.3 Solution of the Killing spinor equations

The solution of the Killing spinor equations in this case is similar to that of the $SU(4) \ltimes \mathbb{R}^8$ case. This is because the conditions conditions that we get for $Sp(2) \ltimes \mathbb{R}^8$ are those of $SU(4) \ltimes \mathbb{R}^8$ but with respect to each I, J and K endomorphisms.

The supersymmetry conditions that arise from the gravitino and dilatino Killing spinor equations can be decomposed in representations of $\mathfrak{sp}(2)$. This is easily done using $\mathfrak{sp}(2) = \mathfrak{so}(5)$ but we shall not pursue this here because of the similarity of this case with that of $SU(4) \ltimes \mathbb{R}^8$. For example, the conditions that arise from the dilatino Killing spinor equation can be written as

$$\begin{aligned}&\partial_+\Phi = 0\ , \quad 2\partial_i\Phi - H_{-+i} = (\theta_I)_i = (\theta_J)_i = (\theta_K)_i\ ,\\ &H_{+ij}(\delta^i{}_m \pm i(I_r)^i{}_m)(\delta^j{}_n \pm i(I_r)^j{}_n) = 0\ ,\\ &H_{ijk}(\delta^i{}_m \pm i(I_r)^i{}_m)(\delta^j{}_n \pm i(I_r)^j{}_n)(\delta^k{}_l \pm i(I_r)^k{}_l) = 0\ ,\end{aligned} \tag{8.41}$$

where θ_I, θ_J and θ_K are the Lee forms of the endomorphisms I, J and K, see (6.22), $(I_r,\, r = 1,2,3) = (I, J, K)$, and $i,j,k,l,m,n = 1,2,3,4,6,7,8,9$. The last three conditions are the vanishing of the (3,0) and (0,3) components of H with respect to I, J and K.

The gravitino Killing spinor equation implies that κ is parallel and so $i_X H = d\kappa$. In addition $\hat\nabla_A(\omega_I)_{ij} = \hat\nabla_A(\omega_J)_{ij} = \hat\nabla_A(\omega_K)_{ij} = 0$ and so the torsion can be expressed in terms of the geometry and the form spinor bilinears ω_I, ω_J and ω_K. The expressions are those that we have given for $SU(4) \ltimes \mathbb{R}^8$ (6.24) but with respect to each of the I, J and K endomorphisms. The only component of the torsion that it is not specified is $H^{\mathbf{10}}_{-ij}$, where the ten-dimensional representation is the adjoint representation of $\mathfrak{sp}(2)$. The metric and torsion can be written in a way similar to that of $SU(4) \ltimes \mathbb{R}^8$ in (6.26).

8.2.4 Special coordinates and a deformation family

As in the $Spin(7) \ltimes \mathbb{R}^8$ and $SU(4) \ltimes \mathbb{R}^8$ cases before, one can introduce coordinates adapted to the parallel vector field X, $X = \partial/\partial u$, and write the metric as in (6.27). The analysis of the construction is similar to the cases we have already investigated and so we shall not pursue this further here. For example, one can introduce a frame $(\mathrm{e}^-, \mathrm{e}^+, e^i)$ adapted to the special coordinates mentioned above as in (5.23).

The spacetime also admits three pairs of integrable distributions, one pair for each of the endomorphisms I, J and K. This is similar to the $SU(4) \ltimes \mathbb{R}^8$ case which we have shown to admit one pair of integrable distributions with respect to the endomorphism I.

The spacetime again has an interpretation as a two parameter family of an eight-dimensional manifold B with an $Sp(2)$-structure. Again for this, one has to introduce a frame (E^-, E^+, E^i) as in (5.31), where (E^+, E^-) are chosen to define an integrable distribution of codimension eight with typical leaf B. There are two cases to consider. If the null vector has non-vanishing rotation then, although B admits an $Sp(2)$-structure, it is not compatible with the induced connection $\hat{\tilde\nabla}$ with torsion. The details are similar to those of the $Spin(7) \ltimes \mathbb{R}^8$ and $SU(4) \ltimes \mathbb{R}^8$ cases, we have already investigated. However, if the rotation of X vanishes, then the above data are compatible, i.e. B is a hyper-complex manifold and all the complex structures are parallel with respect to the induced connection $\hat{\tilde\nabla}$ with torsion. The conditions of the dilatino Killing spinor equation (8.41) also imply that B is conformally balanced. Therefore B is a conformally balanced HKT manifold. The geometric properties of such manifolds have been extensively investigated in the literature, see e.g. [12, 13, 19, 20].

8.2.5 Field Equations

The integrability conditions of the Killing spinor equations imply that if the Bianchi identities of H and F are satisfied, $BH = 0, BF = 0$, then all the field equations are satisfied provided that $E_{--} = 0$, $LH_{-A} = 0$ and $LF_{-} = 0$. This may have been expected because these models are special cases of those with $Spin(7) \ltimes \mathbb{R}^8$- and $SU(4) \ltimes \mathbb{R}^8$-invariant spinors. Examples of backgrounds with $Sp(2) \ltimes \mathbb{R}^8$-invariant Killing spinors have been given in, e.g. [48, 49].

9. $N = 4$ backgrounds

9.1 Backgrounds with $SU(3)$-invariant spinors

As we have explained, without loss of generality the Killing spinors can be chosen as

$$\epsilon_1 = 1 + e_{1234} \ , \quad \epsilon_2 = i(1 - e_{1234}) \ , \quad \epsilon_3 = (e_{15} + e_{2345}) \ , \quad \epsilon_4 = i(e_{15} - e_{2345}) \, . \tag{9.1}$$

Substituting these into the gravitino Killing spinor equation, one finds that the connection $\hat{\nabla}$ takes values in $su(3)$, i.e.

$$\begin{aligned} &\hat{\Omega}_{A,-+} = \hat{\Omega}_{A,-1} = \hat{\Omega}_{A,+1} = \hat{\Omega}_{A,1\bar{1}} = \hat{\Omega}_{A,n}{}^{n} = \hat{\Omega}_{A,np} = 0 \ , \\ &\hat{\Omega}_{A,-n} = \hat{\Omega}_{A,+n} = \hat{\Omega}_{A,1n} = \hat{\Omega}_{A,\bar{1}n} = 0 \ , \qquad m, n, p, \ldots = 2, 3, 4. \end{aligned} \tag{9.2}$$

In addition, the dilatino Killing spinor equation implies the conditions

$$\begin{aligned} \partial_{+}\Phi = \partial_{-}\Phi = \partial_{1}\Phi = \partial_{\bar{1}}\Phi = 0 \ , \\ \partial_{\bar{n}}\Phi - \frac{1}{2} H_{\bar{n}p}{}^{p} = 0 \ , \end{aligned} \tag{9.3}$$

$$\begin{aligned} H_{+1\bar{1}} + H_{+n}{}^{n} = 0 \ , \quad H_{+np} = 0 \ , \quad H_{+\bar{1}\bar{n}} = 0 \ , \\ H_{-1\bar{1}} - H_{-n}{}^{n} = 0 \ , \quad H_{-np} = 0 \ , \quad H_{-1\bar{n}} = 0 \ , \\ H_{1np} = H_{npm} = 0 \ , \\ H_{\bar{1}n}{}^{n} + H_{-+\bar{1}} = 0 \ , \quad H_{1\bar{n}\bar{p}} = 0 \ , \quad H_{\bar{n}1\bar{1}} + H_{-+\bar{n}} = 0 \, . \end{aligned} \tag{9.4}$$

It remains to investigate the restrictions on the geometry of spacetime imposed by the above conditions.

9.2 Spinor bilinears, backgrounds with four isometries and supersymmetry conditions

The conditions (9.2) imply that the holonomy of the connection $\hat{\nabla}$ is contained in $SU(3)$, $\mathrm{hol}(\hat{\nabla}) \subseteq SU(3)$. The form bilinears of the first two Killing spinors have already been computed in the context of supersymmetric backgrounds with $SU(4) \ltimes \mathbb{R}^8$-invariant spinors, and the form bilinears of the first and the third Killing spinor have already been computed in the context of supersymmetric backgrounds with G_2-invariant spinors. The remaining spinor pairs give the one-forms

$$\kappa(\epsilon_1, \epsilon_4) = -\kappa(\epsilon_2, \epsilon_3) = -e^6 \ , \quad \kappa(\epsilon_2, \epsilon_4) = -e^1 \ ,$$

$$\kappa(\epsilon_3, \epsilon_3) = \kappa(\epsilon_4, \epsilon_4) = e^0 + e^5 \ , \tag{9.5}$$

the three-forms

$$\begin{aligned}
\xi(\epsilon_1, \epsilon_4) &= e^0 \wedge e^5 \wedge e^6 - \mathrm{Im}(\hat{\chi}) - e^1 \wedge \hat{\omega} \ , \\
\xi(\epsilon_2, \epsilon_3) &= -e^0 \wedge e^5 \wedge e^6 - \mathrm{Im}(\hat{\chi}) + e^1 \wedge \hat{\omega} \ , \\
\xi(\epsilon_2, \epsilon_4) &= -e^0 \wedge e^1 \wedge e^5 - \mathrm{Re}(\hat{\chi}) + e^6 \wedge \hat{\omega} \ , \\
\xi(\epsilon_3, \epsilon_4) &= (e^0 + e^5) \wedge (e^1 \wedge e^6 + \hat{\omega}) \ ,
\end{aligned} \tag{9.6}$$

and the five-forms

$$\begin{aligned}
\tau(\epsilon_1, \epsilon_4) &= e^0 \wedge e^5 \wedge \mathrm{Im}(\hat{\chi}) + e^1 \wedge e^6 \wedge \mathrm{Re}(\hat{\chi}) + \frac{1}{2} e^6 \wedge \hat{\omega} \wedge \hat{\omega} - \hat{\omega} \wedge e^0 \wedge e^1 \wedge e^5 \ , \\
\tau(\epsilon_2, \epsilon_3) &= e^0 \wedge e^5 \wedge \mathrm{Im}(\hat{\chi}) + e^1 \wedge e^6 \wedge \mathrm{Re}(\hat{\chi}) - \frac{1}{2} e^6 \wedge \hat{\omega} \wedge \hat{\omega} + \hat{\omega} \wedge e^0 \wedge e^1 \wedge e^5 \ , \\
\tau(\epsilon_2, \epsilon_4) &= e^0 \wedge e^5 \wedge \mathrm{Re}(\hat{\chi}) - e^1 \wedge e^6 \wedge \mathrm{Im}(\hat{\chi}) + \frac{1}{2} e^1 \wedge \hat{\omega} \wedge \hat{\omega} - \hat{\omega} \wedge e^0 \wedge e^5 \wedge e^6 \ , \\
\tau(\epsilon_3, \epsilon_3) &= (e^0 + e^5) \wedge \left[-e^1 \wedge \mathrm{Re}(\hat{\chi}) - e^6 \wedge \mathrm{Im}(\hat{\chi}) - \frac{1}{2} \hat{\omega} \wedge \hat{\omega} - \hat{\omega} \wedge e^1 \wedge e^6 \right] \ , \\
\tau(\epsilon_3, \epsilon_4) &= (e^0 + e^5) \wedge [e^1 \wedge \mathrm{Im}(\hat{\chi}) - e^6 \wedge \mathrm{Re}(\hat{\chi})] \ , \\
\tau(\epsilon_4, \epsilon_4) &= -(e^0 + e^5) \wedge \left[-e^1 \wedge \mathrm{Re}(\hat{\chi}) - e^6 \wedge \mathrm{Im}(\hat{\chi}) + \frac{1}{2} \hat{\omega} \wedge \hat{\omega} + \hat{\omega} \wedge e^1 \wedge e^6 \right] .
\end{aligned} \tag{9.7}$$

As we have explained all the form bilinears of the parallel spinors are parallel with respect to the $\hat{\nabla}$ connection. In particular, the one-forms e^-, e^+, e^1 and $\mathrm{e}^{\bar{1}}$ are $\hat{\nabla}$-parallel. Let us denote with X, Y, Z and $\bar{Z}$ the associated vector fields, respectively. In general, the vector space spanned by X, Y, Z and $\bar{Z}$ is not closed under Lie brackets. To see this recall that commutator of two vector field, say X, Y, is given by $i_X i_Y H$ and the supersymmetry conditions do not imply that the component H_{-+k} of H vanishes, and similarly for the other vector fields. This is reminiscent to the situation we have encountered in the backgrounds with G_2-invariant Killing spinors. Again there are many possibilities ranging from requiring X, Y, Z and $\bar{Z}$ to commute to taking the spacetime to be a non-abelian Lorentzian Lie group. We shall not investigate all cases, instead we shall require that the vector space $\mathfrak{h}$ spanned by X, Y, Z and $\bar{Z}$ closes under Lie brackets, i.e. that $\mathfrak{h}$ is a Lie algebra. This in particular implies that

$$H_{abn} = 0 \ , \quad a, b = +, -, 1, \bar{1} \ , \quad n = 2, 3, 4 \, . \tag{9.8}$$

Observe that some of these conditions are already implied by the dilatino Killing spinor equation. The remaining non-vanishing commutators of the vector fields are unconstrained by the conditions of the dilatino Killing spinor equation and so they can span any four-dimensional Lorentzian Lie algebra. The conditions from the gravitino and dilatino Killing spinor equations in the case where $\mathfrak{h}$ is a Lie algebra can be summarized as

$$\hat{\Omega}_{A,ab} = \hat{\Omega}_{A,n}{}^{n} = \hat{\Omega}_{A,an} = \hat{\Omega}_{A,np} = 0 \ , \tag{9.9}$$

and

$$\partial_a \Phi = 0 \ , \quad \partial_{\bar{n}} \Phi - \frac{1}{2} H_{\bar{n}p}{}^{p} = 0$$

$$H_{mnp} = H_{abn} = H_{anp} = 0 \ , \quad \frac{1}{3!}\epsilon_a{}^{bcd} H_{bcd} - iH_{ap}{}^p = 0 \ , \tag{9.10}$$

respectively, where $\epsilon = -i\mathrm{e}^- \wedge \mathrm{e}^+ \wedge \mathrm{e}^1 \wedge \mathrm{e}^{\bar{1}}$. The above supersymmetry conditions have been written in apparent representations of $\mathfrak{so}(3,1) \oplus \mathfrak{su}(3)$.

We have explained that $H_{abn} = 0$ is required by the closure of the Lie brackets of the vector fields. The condition $H_{anp} = 0$ implies that

$$\mathcal{L}_X \hat{\omega} = 0 \ , \quad \mathcal{L}_Y \hat{\omega} = 0 \ , \quad \mathcal{L}_Z \hat{\omega} = 0 \ , \quad \mathcal{L}_{\bar{Z}} \hat{\omega} = 0 \, . \tag{9.11}$$

However, unlike the G_2 case, X, Y, Z and $\bar{Z}$ do not always preserve the $SU(3)$-structure. This is because from the last condition of the dilatino Killing spinor equations the components $H_{ak}{}^k$ of the torsion are not required to vanish, unless $\mathfrak{h}$ is abelian. In particular, we find that

$$\mathcal{L}_{X_a} \hat{\chi} = H_{an}{}^n \hat{\chi} \ , \tag{9.12}$$

where $X_a = (X, Y, Z, \bar{Z})$. Thus the parallel vector fields do not preserve the holomorphic volume form of the $SU(3)$-structure.

9.2.1 Solution of the Killing spinor equations

As for the supersymmetric backgrounds with G_2-invariant spinors, the spacetime M of backgrounds with $SU(3)$-invariant spinors can be interpreted as a principal bundle $M = P(\mathcal{H}, B, \pi)$ equipped with a connection λ, where $\mathcal{H}$ is a Lorentzian group with Lie algebra $\mathfrak{h}$ spanned by the four parallel vector fields, B is the space of orbits of $\mathcal{H}$ in $M = P$, and π is the projection of the principal bundle. The various formulae we have found in (7.1.4) can be extended to the $SU(3)$ case we are investigating here. In particular, the connection on the principal bundle can again be chosen as $\lambda^a = e^a$, where now $a = +, -, 1, \bar{1}$.

Combining the conditions of the gravitino and dilatino Killing spinor equations, we find that the Levi-Civita connection of the spacetime satisfies the conditions,

$$\begin{aligned}
&\Omega_{n,ab} = 0 \ , \quad 2\Omega_{[a,bc]} = H_{abc} \ , \quad \Omega^{\mathbf{20}}_{a,bc} = 0 \ , \\
&2\Omega_{a,n}{}^n = H_{an}{}^n = -\frac{i}{3!}\epsilon_a{}^{bcd} H_{bcd} \ , \quad 2\Omega_{p,n}{}^n = H_{pn}{}^n = -2\partial_p \Phi \ , \\
&\Omega_{a,bn} = 0 \ , \quad \Omega_{n,pa} = 0 \ , \quad 2\Omega_{\bar{n},ap} = H_{\bar{n}ap} \ , \\
&\Omega_{n,pm} = 0 \ , \quad 2\Omega_{\bar{n},mq} = H_{\bar{n}mq} \ , \quad \Omega_{a,np} = 0 \ ,
\end{aligned} \tag{9.13}$$

which we have expressed in terms of $\mathfrak{so}(3,1) \oplus \mathfrak{su}(3)$ representations. Some of above conditions can also be seen as expressing the flux H in terms of the Levi-Civita connection. As we shall see H is determined from the spinor bi-linears. Using the relations (9.13), the torsion free conditions of the frame e^A can be written as

$$\begin{aligned}
de^a &= \frac{1}{2} H^a{}_{bc} e^b \wedge e^c + H^a{}_{n\bar{p}} \mathrm{e}^n \wedge \mathrm{e}^{\bar{p}} \ , \\
d\mathrm{e}^n &= -\Omega_{a,}{}^n{}_p e^a \wedge \mathrm{e}^p + \frac{1}{2} H^n{}_{pa} \mathrm{e}^p \wedge e^a - \Omega_{\bar{p},}{}^n{}_m \mathrm{e}^{\bar{p}} \wedge \mathrm{e}^m \\
&\qquad - \Omega_{p,}{}^n{}_{\bar{m}} \mathrm{e}^p \wedge \mathrm{e}^{\bar{m}} - \Omega_{p,}{}^n{}_m \mathrm{e}^p \wedge \mathrm{e}^m \, .
\end{aligned} \tag{9.14}$$

In terms of principal bundle data, the first torsion free condition above can be interpreted as the Cartan structure equation for the connection λ, i.e.

$$d\lambda^a - \frac{1}{2}H^a{}_{bc}\lambda^b \wedge \lambda^c - \frac{1}{2}H^a{}_{ij}e^i \wedge e^j = 0 \ , \quad i,j,k,l = 2,3,4,7,8,9 \ , \tag{9.15}$$

and so the curvature of λ is

$$\mathcal{F}^a = \frac{1}{2}H^a{}_{ij}e^i \wedge e^j \ . \tag{9.16}$$

In addition the condition (9.10) implies that the curvature $\mathcal{F}^a$ satisfies the Donaldson condition. Note that $\mathcal{F}$ takes values in $\mathfrak{u}(3)$ rather than in $\mathfrak{su}(3)$ because if $\mathfrak{h}$ is not abelian, the complex trace of $\mathcal{F}$ does not vanish. The torsion H of spacetime can be written as in (7.44). Therefore the metric and torsion of spacetime in terms of principal bundle data can be written as

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + \delta_{ij}e^ie^j \\ H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \pi^*\tilde{H} \end{aligned} \tag{9.17}$$

where $\tilde{H}$ is a three form of the base space B horizontally lifted to P.

It remains to investigate the geometry of the base space B. The Riemannian manifold B is equipped with a metric $d\tilde{s}^2$ and a three form $\tilde{H}$ such that $\pi^*d\tilde{s}^2 = \delta_{ij}e^ie^j$. Therefore, one can define a metric connection $\hat{\tilde{\nabla}}$ on B with skew-symmetric torsion. In addition B is equipped with a two-form $\tilde{\omega}$ such that $\hat{\omega} = \pi^*\tilde{\omega}$. This follows from the property of $\hat{\omega}$ to be invariant under $\mathcal{H}$. The two-form $\tilde{\omega}$ is parallel with respect to $\hat{\tilde{\nabla}}$. This follows from $\hat{\nabla}\hat{\omega} = 0$. The associated almost complex structure $\tilde{I}$ of the pair $(d\tilde{s}^2, \tilde{\omega})$ is integrable. This follows from the fact that $\tilde{H}$ is (2,1) and (1,2) with respect to $\tilde{I}$. Therefore B is a KT manifold which is conformally balanced. The latter condition follows from the second equation in the conditions that arise from the dilatino Killing spinor equation in (9.10), i.e. the Lee form can be written as $\tilde{\theta} = 2d\Phi$, where

$$\tilde{\theta} = -\star(\star d\tilde{\omega} \wedge \tilde{\omega}) \ , \tag{9.18}$$

and $d\mathrm{vol}(B) = e^2 \wedge e^3 \wedge e^4 \wedge e^7 \wedge e^8 \wedge e^9$. It is well-known that for such manifolds the torsion three-form is uniquely determined in terms of the complex structure and the metric as

$$\tilde{H} = -i_{\tilde{I}}d\tilde{\omega} = \star d\tilde{\omega} - \star(\tilde{\theta} \wedge \tilde{\omega}) \, . \tag{9.19}$$

It remains to find whether B has a compatible $SU(3)$-structure. There are two cases to consider depending on whether or not $\mathcal{H}$ is abelian.

First suppose that $\mathcal{H}$ is abelian, then $H_{an}{}^n = 0$ and therefore $\hat{\chi}$ is invariant under $\mathcal{H}$. In such a case B admits a compatible $SU(3)$-structure, i.e. there is a (3,0)-form $\tilde{\chi}$ on B such that it is parallel with respect to $\hat{\tilde{\nabla}}$. Thus $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(3)$. Therefore B is a conformally balanced KT manifold with a compatible $SU(3)$-structure. An analysis of the geometry of these manifolds in terms of G-structures can be found in [43, 11]. The covariant derivatives $\tilde{\nabla}\tilde{\omega}$ and $\tilde{\nabla}\tilde{\chi}$ can be decomposed in terms of five irreducible $SU(3)$ representations as in the

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case of eight-dimensional manifolds with $SU(4)$-structures that we have already described. Since B is complex $W_1 = W_2 = 0$. In addition W_4 and W_5, which can be represented by the Lee forms $\theta_{\tilde{\omega}} = \tilde{\theta}$ given in (9.18) and

$$\theta_{\mathrm{Re}\tilde{\chi}} = -\frac{1}{2} \star (\star d\mathrm{Re}\tilde{\chi} \wedge \mathrm{Re}\tilde{\chi}), \tag{9.20}$$

respectively, are related as

$$\theta_{\tilde{\omega}} = \theta_{\mathrm{Re}\tilde{\chi}} = 2d\Phi\,. \tag{9.21}$$

Next suppose that $\mathcal{H}$ is not abelian. There are three distinct four-dimensional non-abelian Lorentzian Lie algebras. This is because the structure constants of such Lie algebra are dual to a vector in four-dimensional Minkowski space. Since the generators of the Lie algebra are determined up to a Lorentz transformation, there are three types of Lie algebras depending on whether the vector is timelike, spacelike or null. These are isomorphic to $\mathbb{R} \oplus \mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$ or $\mathfrak{so}(2) \oplus_s \mathfrak{h}_2(\mathbb{R})$, respectively, where $\mathfrak{h}_2$ is the Heisenberg algebra of dimension three. In this case, $\hat{\chi}$ is not invariant under $\mathcal{H}$. As a result although there still exists a $\tilde{\chi}$ on B which horizontally lifts to $\hat{\chi}$, $\tilde{\chi}$ is a section of $\Lambda^{3,0} \otimes L$, where $L = P \times_\rho \mathbb{C}$ is a line bundle over B associated to the principal bundle P. The representation[15] ρ is induced from (9.12). Therefore $\tilde{\chi}$ is a tensorial form of degree three associated with the representation ρ in the terminology of [45]. The structure that it is associated with such a form is reminiscent of a $Spin_c$-structure and so we shall say that B admits an $SU_c(3)$-structure but not an $SU(3)$ one as one might have been expecting. Therefore B is a conformally balanced KT manifold. The geometry of these manifolds can also be examined using the Gray-Hervella classes [22]. It turns out that $W_1 = W_2 = 0$, because B is complex, and W_4 which is represented by the Lee form $\tilde{\theta}$ defined in (9.18).

To examine some other aspects of the geometry of spacetime, we consider the one-forms $\{\mathrm{e}^n, \mathrm{e}^{\bar{n}}\}$, $\{\mathrm{e}^n, \mathrm{e}^1, \mathrm{e}^-,\}$ and $\{e^a, \mathrm{e}^n\}$. These span integrable distributions of co-dimensions four, five and three, respectively.[16] The first distribution is associated with the principal bundle structure of the spacetime which we have already investigated. The second distribution implies that the space admits a certain Lorentzian complex structure, i.e. the spacetime is a "Lorentzian"-holomorphic manifold. Observe that $\{\mathrm{e}^{\bar{n}}, \mathrm{e}^{\bar{1}}, \mathrm{e}^-,\}$ is also an integrable distribution. The third distribution is related to the property of B to be a complex manifold.

To summarize the geometry of backgrounds with $SU(3)$ invariant spinors, we have found that the spacetime is (locally) a principal bundle $M = P(\mathcal{H}, B, \pi)$ equipped with a connection λ. The fibre is a four-dimensional Lorentzian Lie group and the curvature $\mathcal{F}$ of the connection λ satisfies the Donaldson condition. The geometry of the base space B depends on whether or not $\mathcal{H}$ is abelian. If $\mathcal{H}$ is abelian, then the base space B is a balanced KT manifold with an $SU(3)$-structure. If $\mathcal{H}$ is not abelian, then B is a balanced KT manifold equipped with a bundle L associated to P and a section $\tilde{\chi}$ of $\Lambda^{3,0} \otimes L$. The metric and torsion of spacetime in terms of principal bundle data are given in (9.17), and the dilaton Φ is a function of B.

[15] We assume that ρ is not trivial. If it is trivial, then the analysis reduces to that of the abelian case.

[16] There are other integrable distributions, i.e. the one spanned by the one-forms $\{\mathrm{e}^n\}$.

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As in the G_2 case, one can compute dH to find

$$dH = \eta_{ab}\, \mathcal{F}^a \wedge \mathcal{F}^b + \pi^* \, d\tilde{H} \,. \tag{9.22}$$

The first term in the right-hand-side of dH is a representative of the first Pontrjagin class of the principal bundle P. If one requires that $dH = 0$, then the representative of the first Pontrjagin class of P must cancel against a form on B. Of course if P is a globally defined principal bundle over B and one imposes the condition $dH = 0$, then the first Pontrjagin form is exact and therefore the first Pontrjagin class of the principal bundle should vanish.

9.2.2 Field equations and examples

An investigation of the integrability conditions of the Killing spinor equations imply all the field equations are satisfied provided that one imposes the Bianchi identities $BH = BF = 0$. This may have been expected because this class of backgrounds is a special case of those with $N = 2$ supersymmetry and G_2-invariant spinors.

In the absence of a gauge field, the only Bianchi identity that has to be imposed is that of H which has been computed in (9.22). As an example, one can take $\mathcal{H}$ to be abelian. Then, one can write $\lambda^a = dx^a + A^a$ and so we have

$$\begin{aligned} ds^2 &= \eta_{ab}(dx^a + A^a)(dx^b + A^b) + \delta_{ij} e^i e^j \;, \\ H &= \eta_{ab}(dx^a + A^a) \wedge dA^b + H' \;, \end{aligned} \tag{9.23}$$

where $\mathcal{F} = dA$ the curvature of the connection takes values in $\mathfrak{su}(3)$, i.e. it satisfies the Donaldson condition. If in addition we require that $dH = 0$, then

$$H = \eta_{ab} dx^a \wedge dA^b + H_B \tag{9.24}$$

after choosing $H' = -\eta_{ab} A^a dA^b + H_B$, where H_B is a three-form on B such that $dH_B = 0$. Within a brane interpretation of these solutions, the connections A^a can be thought of as a rotation and wrapping.

A special case of this example is whenever the only non-vanishing rotation and wrapping is a along a null direction. In this case, the Chern-Simons form contribution vanishes. Thus one can set $\tilde{H} = H_B$. Such kind of solutions have been consider before in [47]. The metric and torsion are

$$\begin{aligned} ds^2 &= 2dv(du + A) + dx^2 + dy^2 + \delta_{ij} e^i e^j \;, \\ H &= dv \wedge dA + H_B \,. \end{aligned} \tag{9.25}$$

In such a case, the base space B is a integrable, conformally balanced strong KT Riemannian manifold such that $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(3)$ and $\tilde{H}$ is closed. An example of such manifold was found in [50] and has been used as gravitational dual to $N = 1$ Yang-Mills theory in four-dimensions [51]. The geometry of these models has been investigated in [7]. It is remarkable that the six-dimensional manifold is also a principal bundle.

9.3 Backgrounds with $(SU(2)\times SU(2))\ltimes\mathbb{R}^8$-invariant spinors

9.3.1 Supersymmetry conditions

As we have already explained in 4.1, one can always arrange without loss of generality such that the Killing spinors are

$$\epsilon_1 = 1 + e_{1234}\ , \quad \epsilon_2 = i(1-e_{1234})\ , \quad \epsilon_3 = e_{12}-e_{34}\ , \quad \epsilon_4 = i(e_{12}+e_{34})\,. \tag{9.26}$$

The gravitino Killing spinor equation implies that the connection $\hat\nabla$ takes values in $(su(2)\oplus su(2))\oplus_s\mathbb{R}^8$, i.e. that

$$\begin{aligned}
\hat\Omega_{A,\alpha\beta} &= \hat\Omega_{A,np} = 0\ , \quad \hat\Omega_{A,\alpha}{}^{\alpha} = \hat\Omega_{A,n}{}^{n} = 0\ ,\\
\hat\Omega_{A,\alpha n} &= \hat\Omega_{A,\bar\alpha n} = 0\ ,\\
\hat\Omega_{A,+B} &= = 0\ , \qquad\qquad \alpha,\beta,\ldots = 1,2\ , \quad n,p,\ldots = 3,4\,.
\end{aligned} \tag{9.27}$$

The conditions that arise from the dilatino Killing spinor equation are

$$\partial_+\Phi = 0\ , \quad \partial_{\bar\alpha}\Phi - \frac{1}{2}H_{\bar\alpha\beta}{}^{\beta} - \frac{1}{2}H_{-+\bar\alpha} = 0\ , \quad \partial_{\bar n}\Phi - \frac{1}{2}H_{\bar n p}{}^{p} - \frac{1}{2}H_{-+\bar n} = 0\ , \tag{9.28}$$

$$H_{\alpha\beta n} = H_{\alpha np} = 0\ , \quad H_{\bar n\alpha\beta} = H_{\bar\alpha np} = 0\ , \quad H_{n\alpha}{}^{\alpha} = H_{\alpha n}{}^{n} = 0\ , \tag{9.29}$$

$$H_{+\alpha\beta} = H_{+\alpha}{}^{\alpha} = H_{+np} = H_{+n}{}^{n} = H_{+\alpha n} = H_{+\alpha\bar n} = 0\,. \tag{9.30}$$

It remains to investigate the restrictions on the geometry of the spacetime that are implied by the above conditions.

9.3.2 Geometry, holonomy and spinor bilinears

As in the previous null cases, the conditions that arise from the gravitino Killing spinor equation (9.27) imply that $\mathrm{hol}(\hat\nabla)\subseteq(SU(2)\times SU(2))\ltimes\mathbb{R}^8$. To give a further insight into the geometry, one can compute the form bilinears of the Killing spinors. This has already been done for the pairs of the first three Killing spinors. Therefore, it remains to compute the forms of the pairs (ϵ_1,ϵ_4), (ϵ_2,ϵ_4), (ϵ_3,ϵ_4) and (ϵ_4,ϵ_4). In particular, we find the one form

$$\kappa(\epsilon_4,\epsilon_4) = e^0 - e^5\ , \tag{9.31}$$

the three-forms

$$\begin{aligned}
\xi(\epsilon_1,\epsilon_4) &= -(e^0-e^5)\wedge\mathrm{Im}(\chi_1+\chi_2)\ ,\\
\xi(\epsilon_2,\epsilon_4) &= -(e^0-e^5)\wedge\mathrm{Re}(\chi_1+\chi_2)\ ,\\
\xi(\epsilon_3,\epsilon_4) &= -(e^0-e^5)\wedge(\omega_1-\omega_2)\ ,
\end{aligned} \tag{9.32}$$

and the five-forms

$$\begin{aligned}
\tau(\epsilon_1,\epsilon_4) &= -(e^0-e^5)\wedge(\omega_1\wedge\mathrm{Re}(\chi_2)+\omega_2\wedge\mathrm{Re}(\chi_1))\ ,\\
\tau(\epsilon_2,\epsilon_4) &= (e^0-e^5)\wedge(\omega_1\wedge\mathrm{Im}(\chi_2)+\omega_2\wedge\mathrm{Im}(\chi_1))\ ,\\
\tau(\epsilon_3,\epsilon_4) &= -(e^0-e^5)\wedge\mathrm{Im}(\chi_1^*\wedge\chi_2)\ ,
\end{aligned}$$

$$\tau(\epsilon_4,\epsilon_4) = -(e^0 - e^5)\wedge(\mathrm{Re}(\chi_1^*\wedge\chi_2) + \frac{1}{2}(\omega_1-\omega_2)\wedge(\omega_1-\omega_2))\ , \tag{9.33}$$

where $\omega_1 = -(e^1\wedge e^6 + e^2\wedge e^7)$, $\omega_2 = -(e^3\wedge e^8 + e^4\wedge e^9)$, $\chi_1 = (e^1+ie^6)\wedge(e^2+ie^7)$ and $\chi_2 = (e^3+ie^8)\wedge(e^4+ie^9)$. All the form bilinears α constructed from the Killing spinors are parallel with respect to the connection $\hat\nabla$,

$$\hat\nabla\alpha = 0\,. \tag{9.34}$$

Moreover as expected, (9.30) implies that $i_X H$ takes values in $(\mathfrak{su}(2)\oplus\mathfrak{su}(2))\oplus_s\mathbb{R}^8$, where X is the associated parallel vector field to κ. This in turn gives that

$$\mathcal{L}_X\alpha = 0\ , \tag{9.35}$$

where again α stands for any form Killing spinor bilinear. Since X is Killing, $\mathcal{L}_X H = 0$ ($dH=0$) and $\mathcal{L}_X\Phi = 0$, X leaves all the geometry of spacetime including the $(SU(2)\times SU(2))\ltimes\mathbb{R}^8$-structure invariant.

9.3.3 Solution of the Killing spinor equations

The solution of the Killing spinor equations is similar to that of the $SU(4)\ltimes\mathbb{R}^8$ case. The supersymmetry conditions of both the gravitino and dilatino Killing spinor equations have been decomposed in terms of $SU(2)\times SU(2)$ representations in an apparent way. The minimal set of covariantly constant forms that characterizes the conditions (9.27) that arise from the gravitino Killing spinor equation are

$$\kappa = \mathrm{e}^-\ ,\quad \xi_I = \mathrm{e}^-\wedge\omega_I\ ,\quad \xi_J = \mathrm{e}^-\wedge\omega_J\ ,\quad \xi_1 = \mathrm{e}^-\wedge\chi_1\ ,\quad \xi_2 = \mathrm{e}^-\wedge\chi_2\,. \tag{9.36}$$

where

$$\omega_I = \omega_1+\omega_2\ ,\quad \omega_J = \omega_1-\omega_2\,. \tag{9.37}$$

In particular, if we denote the forms (9.36) collectively with β, then the conditions that arise from the gravitino Killing spinor equation are equivalent to

$$\hat\nabla\beta = 0\,. \tag{9.38}$$

Note that the endomorphisms I, J of the tangent bundle of the spacetime commute, i.e. $IJ = JI$. The forms (ω_1,χ_1) and (ω_2,χ_2) are associated with an $SU(2)\times SU(2)$-structure.

The conditions (9.29) imply that H is a (2,1) and (1,2) form with respect to both I and J, i.e. the (3,0) and (0,3) components with respect to both I, J vanish. The last two conditions in (9.28) can be rewritten as

$$2\partial_\alpha\Phi = (\theta_1)_\alpha\ ,\quad 2\partial_n\Phi = (\theta_2)_n\ , \tag{9.39}$$

where θ_1 and θ_2 are the Lee forms of the endomorphisms I_1 and I_2 associated with ω_1 and ω_2, respectively, see (6.22). The conditions (9.30) imply that H_{+ij}, $i,j = 1,2,3,4,6,7,8,9$ takes values in $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$.

The supersymmetry conditions can be solved for the fluxes. It is easy to see that the expressions for $H_{-\alpha\beta}$ and $H_{\alpha\beta\bar{\gamma}}$, and H_{-np} and $H_{nm\bar{q}}$ can be given as in (6.24) but now with respect to the endomorphisms I_1 and I_2 associated to the forms ω_1 and ω_2, respectively. Similarly $H_{-\alpha}{}^{\alpha}$ and $H_{-n}{}^{n}$ can be expressed as in (6.25) but now with respect to χ_1 and χ_2, respectively. Therefore all H fluxes are determined in terms of the spinor bilinears and the metric of spacetime apart from the component $H^{\mathbf{6}}_{-ij}$ which takes values in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The metric and torsion of the spacetime can be written as

$$\begin{aligned}
ds^2 &= 2\mathrm{e}^-\mathrm{e}^+ + 2\delta_{\alpha\bar{\beta}}\mathrm{e}^\alpha\mathrm{e}^{\bar{\beta}} + 2\delta_{m\bar{n}}\mathrm{e}^m\mathrm{e}^{\bar{n}} \\
H &= \mathrm{e}^+ \wedge d\kappa - [\frac{1}{2}(I_1)^m{}_i\nabla_-(\omega_1)_{mj} + \frac{1}{2}(I_2)^m{}_i\nabla_-(\omega_2)_{mj} + (I_1)^m{}_i\nabla_-(\omega_2)_{mj} \\
&\quad + (I_2)^m{}_i\nabla_-(\omega_1)_{mj}]e^- \wedge e^i \wedge e^j - \frac{1}{8}\,\mathrm{Im}((\bar{\chi}_1)^{\alpha\beta}\nabla_-(\chi_1)_{\alpha\beta})\, e^- \wedge \omega_1 \\
&\quad - \frac{1}{8}\,\mathrm{Im}((\bar{\chi}_2)^{mn}\nabla_-(\chi_2)_{mn})e^- \wedge \omega_2 + \frac{1}{2}H^{\mathbf{6}}_{-ij}e^- \wedge e^i \wedge e^j \\
&\qquad + \frac{1}{3!}H_{ijk}e^i \wedge e^j \wedge e^k\ ,
\end{aligned} \tag{9.40}$$

where $i,j,k,l = 1,2,3,4,6,7,8,9,$ H_{ijk} is determined in terms of ω_1, ω_2, I_1 and I_2 as has been explained above.

9.3.4 Local coordinates, distributions and a deformation family

One can adapt coordinates along the null parallel vector field X and write the metric of the spacetime as in (5.22). Then one can adapt a frame $(\mathrm{e}^-, \mathrm{e}^+, \mathrm{e}^\alpha, \mathrm{e}^{\bar{\alpha}}, \mathrm{e}^n, \mathrm{e}^{\bar{n}})$ in a way similar to that in (5.23). The spacetime admits various integrable distributions. Apart from the distributions of co-dimension five spanned by $(\mathrm{e}^-, \mathrm{e}^\alpha, \mathrm{e}^n)$ and $(\mathrm{e}^-, \mathrm{e}^{\bar{\alpha}}, \mathrm{e}^{\bar{n}})$ which are analogues to those of the $SU(4)\ltimes\mathbb{R}^8$ backgrounds, there are also integrable distributions of codimension three spanned by $(\mathrm{e}^-, \mathrm{e}^\alpha, \mathrm{e}^n, \mathrm{e}^{\bar{n}})$ and $(\mathrm{e}^-, \mathrm{e}^\alpha, \mathrm{e}^{\bar{\beta}}, \mathrm{e}^n)$. This can be easily seen using the torsion free conditions of the frame $(\mathrm{e}^-, \mathrm{e}^+, \mathrm{e}^\alpha, \mathrm{e}^{\bar{\alpha}}, \mathrm{e}^n, \mathrm{e}^{\bar{n}})$.

The spacetime has the interpretation as a two parameter family of an eight-dimensional manifold with an $SU(2)\times SU(2)$-structure. This is done by adapting a frame (E^-, E^+, E^i) similar to that in (5.31). The one-forms (E^-, E^+) span a codimension eight integrable distribution with leaves the eight-dimensional manifold B. There are two cases to consider. In the generic case, and in particular if the rotation of the null vector field does not vanish, $d\kappa = d\mathrm{e}^- \neq 0$, B admits an $SU(2) \times SU(2)$-structure which is *not* compatible with the induced connection $\hat{\tilde{\nabla}}$ with torsion $\tilde{H} = H|_B$. The explanation for this has been presented in detail for the $Spin(7) \ltimes \mathbb{R}^8$ case and it will not be repeated here. However, if $d\kappa = d\mathrm{e}^- = 0$, then B admits an $SU(2) \times SU(2)$-structure which is compatible with the induced connection $\hat{\tilde{\nabla}}$. In such a case, B is a conformally balanced eight-dimensional manifold equipped with (i) metric connection $\hat{\tilde{\nabla}}$ with torsion given by a three-form $\tilde{H}$, (ii) two commuting complex structures $\tilde{I}$ and $\tilde{J}$, $\tilde{I}\tilde{J} = \tilde{J}\tilde{I}$, which are the restrictions of I, J, such that $\hat{\tilde{\nabla}}\tilde{I} = \hat{\tilde{\nabla}}\tilde{J} = 0$ and (iii) $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(2)\times SU(2)$. The manifold B is conformally balanced because the Lee forms satisfy $\tilde{\theta}_{\tilde{I}} = \tilde{\theta}_{\tilde{J}} = 2d\tilde{\Phi}$ as required by the supersymmetry conditions of the dilatino Killing spinor equation (9.28). Note that B admits an integrable product structure $\Pi = \tilde{I}\tilde{J}$, i.e. $\Pi^2 = 1$. The geometry of B can be also analyzed using

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G-structures but this is very similar to the $SU(n)$ cases we have already examined and we shall not pursue this further here.

The geometric properties of manifolds equipped with two commuting complex structures compatible with a metric connection with torsion have been investigated before in the physics literature in the context of sigma models with (2,2) worldvolume supersymmetry [1, 3]. In particular, special coordinates have been introduced and the local expression for the metric has been given. In addition the simultaneous integrability properties of the complex structures and the product structure have been examined in detail. As in the previews null cases, the integrability conditions of the Killing spinor imply that all field equations are satisfied provided that the Bianchi identities are imposed and one requires that $E_{--}=0$, $LH_{-A}=0$ and $LF_{-}=0$.

10. $N=8$ backgrounds

10.1 Backgrounds with $SU(2)$-invariant spinors

10.1.1 Supersymmetry conditions

One can choose without loss of generality the Killing spinors, see section 4.1, as

$$\begin{aligned}&\epsilon_1=1+e_{1234}\ ,\quad \epsilon_2=i(1-e_{1234})\ ,\quad \epsilon_3=e_{12}-e_{34}\ ,\quad \epsilon_4=i(e_{12}+e_{34})\ ,\\ &\epsilon_5=e_{15}+e_{2345}\ ,\ \epsilon_6=i(e_{15}-e_{2345})\ ,\ \epsilon_7=e_{52}+e_{1345}\ ,\ \epsilon_8=i(e_{52}-e_{1345})\,.\end{aligned}\tag{10.1}$$

Substituting them in to the gravitino Killing spinor equation, one finds that the connection $\hat\nabla$ takes values in $su(2)$, i.e.

$$\begin{aligned}&\hat\Omega_{A,+\alpha}=\hat\Omega_{A,-\alpha}=\hat\Omega_{A,-+}=\hat\Omega_{A,\alpha\beta}=\hat\Omega_{A,\alpha\bar\beta}=\hat\Omega_{A,\alpha n}=\hat\Omega_{A,\alpha\bar n}=0\ ,\\ &\hat\Omega_{A,+n}=\hat\Omega_{A,-n}=0\ ,\quad \hat\Omega_{A,np}=\hat\Omega_{A,n}{}^{n}=0\ ,\quad \alpha,\beta=1,2\ ,\quad n,p=3,4\,.\end{aligned}\tag{10.2}$$

Similarly, the conditions that arise from the dilatino Killing spinor equation equation are

$$\begin{aligned}&\partial_+\Phi=\partial_-\Phi=\partial_\alpha\Phi=0\ ,\quad \partial_{\bar n}\Phi-\frac{1}{2}H_{\bar np}{}^{p}=0\ ,\\ &H_{\bar\alpha\beta}{}^{\beta}+H_{-+\bar\alpha}=0\ ,\quad H_{\alpha n}{}^{n}=0\ ,\quad H_{-+n}=0\ ,\\ &H_{\alpha\beta n}=H_{\alpha np}=0\ ,\quad H_{\bar n\alpha\beta}=H_{\bar\alpha np}=0\ ,\quad H_{n\alpha\bar\beta}=0\ ,H_{n\alpha}{}^{\alpha}=0\ ,\\ &H_{-n}{}^{n}=0\ ,\quad H_{-\alpha n}=0\ ,\quad H_{-\bar n\bar p}=0\ ,\quad H_{-\alpha\bar n}=0\ ,\quad H_{-1\bar 2}=0\ ,\quad H_{-1\bar 1}=H_{-2\bar 2}\ ,\\ &H_{+\alpha\beta}=0\ ,\quad H_{+\alpha}{}^{\alpha}=0\ ,\quad H_{+np}=0\ ,\quad H_{+n}{}^{n}=0\ ,\quad H_{+\alpha n}=0\ ,\quad H_{+\alpha\bar n}=0\,.\end{aligned}\tag{10.3}$$

We shall next investigate the restrictions that these conditions above impose on the geometry of spacetime.

10.1.2 Geometry and form bilinears

The gravitino Killing spinor equation implies that the holonomy of $\hat\nabla$ is contained in $SU(2)$, $\mathrm{hol}(\hat\nabla)\subseteq SU(2)$. Several of the Killing spinor form bilinears have been computed in previous cases. The remaining pairs can also easily be computed. This is because we have found the form spinor bilinears of all possible types of spinors, see A.2.6. As a result, the forms of the remaining pairs are given from those we have computed already by an appropriately relabeling of indices. However, we shall not explicitly give the result because it is not particularly enlightening.

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All the forms that arise form the Killing spinor bilinears are parallel with respect to $\hat{\nabla}$. A basis in the ring of parallel forms is $\mathrm{e}^-, \mathrm{e}^+, e^1, e^6, e^2, e^7, \omega_2$ and χ_2, where

$$\omega_2 = -(e^3 \wedge e^8 + e^4 \wedge e^9) \ , \quad \chi_2 = (e^3 + ie^8) \wedge (e^4 + ie^9) \, . \tag{10.4}$$

In particular this implies that the one-forms $\mathrm{e}^-, \mathrm{e}^+, \mathrm{e}^\alpha, \mathrm{e}^{\bar{\alpha}}$, $\mathrm{e}^\alpha = \frac{1}{\sqrt{2}}(e^\alpha + ie^{5+\alpha})$, are parallel and so the associated vectors fields $X, Y, Z_\alpha, Z_{\bar{\alpha}}$ are Killing. Unlike the cases of backgrounds with G_2- and $SU(3)$-invariant spinors, the vector space $\mathfrak{h} = \mathrm{Span}(X, Y, Z_\alpha, Z_{\bar{\alpha}})$ *closes under Lie brackets*. This is a consequence of the conditions of the dilatino Killing spinor equations and in particular the vanishing of the components

$$H_{abn} = 0 \ , \quad a, b = +, -, \alpha, \bar{\alpha} \ , \quad n = 3, 4 \tag{10.5}$$

of the NS⊗NS three-form field strength. The Lie algebra $\mathfrak{h}$ is not arbitrary but rather constrained by supersymmetry. In particular, the structure constants satisfy the conditions

$$\begin{aligned} &H_{\bar{\alpha}\beta}{}^\beta + H_{-+\bar{\alpha}} = 0 \ , \quad H_{-1\bar{2}} = 0 \ , \quad H_{-1\bar{1}} = H_{-2\bar{2}} \ , \\ &H_{+\alpha\beta} = 0 \ , \quad H_{+\alpha}{}^\alpha = 0 \ , \quad \alpha, \beta = 1, 2 \ , \end{aligned} \tag{10.6}$$

of the dilatino Killing spinor equation. Observe that the structure constants of $Sl(2, \mathbb{R}) \times SU(2)$ whose lie algebras are spanned by $sl(2, \mathbb{R}) = \mathbb{R} < e_-, e_+, e_1 = \frac{1}{\sqrt{2}}(e_1 + e_{\bar{1}}) >$ and $su(2) = \mathbb{R} < e_2, e_{\bar{2}}, e_6 = \frac{1}{i\sqrt{2}}(e_{\bar{1}} - e_1) >$ satisfy these conditions provided one identifies their structure constants as in the first condition of (10.6). The remaining conditions of the dilatino Killing spinor equation, apart from those involving the dilaton, imply that

$$\mathcal{L}_X \omega_2 = 0 \ , \quad \mathcal{L}_Y \omega_2 = 0 \ , \quad \mathcal{L}_{Z_\alpha} \omega_2 = 0 \ , \quad \mathcal{L}_{Z_{\bar{\alpha}}} \omega_2 = 0 \ , \tag{10.7}$$

and

$$\mathcal{L}_X \chi_2 = 0 \ , \quad \mathcal{L}_Y \chi_2 = 0 \ , \quad \mathcal{L}_{Z_\alpha} \chi_2 = 0 \ , \quad \mathcal{L}_{Z_{\bar{\alpha}}} \chi_2 = 0 \, . \tag{10.8}$$

The conditions involving the dilaton imply that Φ is invariant under the action of $\mathfrak{h}$ on the spacetime and that $\partial_n \Phi$ is related to the torsion H.

The conditions on the geometry of supersymmetric backgrounds with $SU(2)$ invariant spinors can be summarized as follows: The gravitino Killing spinor equation gives

$$\hat{\Omega}_{A,ab} = 0 \ , \quad \hat{\Omega}_{A,aB} = 0 \ , \quad \hat{\Omega}_{A,np} = 0 \ , \quad \hat{\Omega}_{A,n}{}^n = 0 \ , \quad a = -, +, \alpha, \bar{\alpha} \ , \tag{10.9}$$

and the dilatino Killing spinor equation gives

$$\begin{aligned} &\partial_a \Phi = 0 \ , \quad \partial_{\bar{n}} \Phi - \frac{1}{2} H_{\bar{n}p}{}^p = 0 \ , \\ &H_{anp} = 0 \ , \quad H_{an}{}^n = 0 \ , \quad H_{nab} = 0 \\ &H_{\bar{\alpha}\beta}{}^\beta + H_{-+\bar{\alpha}} = 0 \ , \quad H_{-1\bar{2}} = 0 \ , \\ &H_{-1\bar{1}} = H_{-2\bar{2}} \ , \quad H_{+\alpha\beta} = 0 \ , \quad H_{+\alpha}{}^\alpha = 0 \, . \end{aligned} \tag{10.10}$$

This concludes the discussion of the supersymmetry conditions.

Combining the conditions of the gravitino and dilatino Killing spinor equations, we find that the Levi-Civita connection of the spacetime satisfies the conditions,

$$\begin{aligned}
&\Omega_{k,ab}=0\;,\quad 2\Omega_{[a,bc]}=H_{abc}\;,\quad \Omega^{\mathbf{70}}_{a,bc}=0\;,\\
&2\Omega_{a,n}{}^{n}=H_{an}{}^{n}=0\;,\quad 2\Omega_{p,n}{}^{n}=H_{pn}{}^{n}=-2\partial_p\Phi\;,\\
&\Omega_{a,bn}=0\;,\quad \Omega_{n,pa}=0\;,\quad 2\Omega_{\bar n,ap}=H_{\bar n ap}\;,\\
&\Omega_{n,pm}=0\;,\quad 2\Omega_{\bar n,mq}=H_{\bar n mq}\;,\quad \Omega_{a,pn}=0\;,
\end{aligned}\tag{10.11}$$

which we have expressed in terms of $\mathfrak{so}(5,1)\oplus\mathfrak{su}(2)$ representations. Some of above conditions can been seen as expressing the flux H in terms of the Levi-Civita connection. In this case, we shall show that H is determined by the metric and the form spinor bilinears.

10.1.3 Solution of the Killing spinor equations

As for the supersymmetric backgrounds with G_2- and $SU(3)$-invariant spinors, the spacetime M of backgrounds with $SU(2)$-invariant spinors can be interpreted as a principal bundle $M=P(\mathcal{H},B,\pi)$ equipped with a connection λ, where $\mathcal{H}$ is a Lorentzian group with Lie algebra $\mathfrak{h}$ spanned by the six parallel vector fields and π is the projection of the principal bundle. Unlike the G_2 and $SU(3)$ cases, the group $\mathcal{H}$ is not arbitrary but its structure constants satisfy the (10.6). The connection is $\lambda^a=e^a$. The Cartan structure equations for λ can be found by considering the torsion free conditions of the frame. Using the relations (10.9) and (10.10), the torsion free conditions of the frame e^A can be written as

$$\begin{aligned}
de^a&=\frac{1}{2}H^a{}_{bc}e^b\wedge e^c+H^a{}_{n\bar p}\,\mathrm{e}^n\wedge\mathrm{e}^{\bar p}\;,\\
de^n&=-\Omega_{a,}{}^{n}{}_{p}e^a\wedge\mathrm{e}^p+\frac{1}{2}H^n{}_{pa}\mathrm{e}^p\wedge e^a-\Omega_{\bar p,}{}^{n}{}_{m}\mathrm{e}^{\bar p}\wedge\mathrm{e}^m\\
&\qquad-\Omega_{p,}{}^{n}{}_{\bar m}\mathrm{e}^p\wedge\mathrm{e}^{\bar m}-\Omega_{p,}{}^{n}{}_{m}\mathrm{e}^p\wedge\mathrm{e}^m\,.
\end{aligned}\tag{10.12}$$

The first condition is interpreted as the Cartan structure equation and so the curvature of the connection λ is

$$\mathcal{F}^a{}_{ij}=\frac{1}{2}H^a{}_{ij}\;,\quad i,j=3,4,8,9\,.\tag{10.13}$$

The conditions (10.10) imply that $\mathcal{F}^a$ is self-dual, i.e. it takes values in the $\mathfrak{su}(2)\subset\mathfrak{so}(4)$. The metric of the spacetime and the torsion H in terms of principal bundle data can be written as

$$\begin{aligned}
ds^2&=\eta_{ab}\lambda^a\lambda^b+\delta_{ij}e^ie^j\;,\\
H&=\frac{1}{3}\eta_{ab}\lambda^a\wedge d\lambda^b+\frac{2}{3}\eta_{ab}\lambda^a\wedge\mathcal{F}^b+\pi^*\tilde H\,.
\end{aligned}\tag{10.14}$$

It remains to investigate the geometry of the four-dimensional base space B. As in the G_2 and $SU(3)$ cases, the base space B is a Riemannian manifold equipped with a metric $d\tilde s^2$ and a three-form $\tilde H$, and so with a metric connection $\tilde{\hat\nabla}$ with three-form torsion. In addition, since both ω_2 and χ_2 are invariant under $\mathcal{H}$, the base space B is also equipped with two two-form $\tilde\omega_2$ and $\tilde\chi_2$. Both $\tilde\omega_2$ and $\tilde\chi_2$ are parallel with respect to $\tilde{\hat\nabla}$. The almost complex structure $\tilde{I}$ associated with the pair $(d\tilde{s}^2, \tilde{\omega}_2)$ is integrable, and $\tilde{\chi}$ is (2,0) with respect $\tilde{I}$. Therefore, B is a conformally balanced HKT manifold. The conformal balance condition follows from the second equation in (10.10) and $\tilde{\theta} = 2d\Phi$, where $\tilde{\theta}$ is the Lee form of $\tilde{\omega}_2$ defined as in (6.22).

As in the $SU(3)$ case, the spacetime admits various distributions spanned by the one-forms $\{\mathrm{e}^n, \mathrm{e}^{\bar{n}}\}$, $\{\mathrm{e}^n, \mathrm{e}^-, \mathrm{e}^1, \mathrm{e}^2\}$ and $\{e^a, \mathrm{e}^n\}$. These are *integrable* distributions of co-dimensions six, five and two, respectively. The first integrable distribution is that of the principal bundle structure that we have already investigated. To show that $\{\mathrm{e}^n, \mathrm{e}^-, \mathrm{e}^1, \mathrm{e}^2\}$ span an integrable distribution, one also needs the conditions (10.6) on the structure constants H_{abc} imposed by supersymmetry. Note that there is another co-dimension five distribution spanned by the one-forms $\{\mathrm{e}^{\bar{n}}, \mathrm{e}^+, \mathrm{e}^{\bar{1}}, \mathrm{e}^2\}$. Both codimension five distributions imply that the spacetime admits a certain "Lorentzian" complex structure, i.e. M is a "Lorentzian-holomorphic" space, in two different ways. The distribution spanned by $\{e^a, \mathrm{e}^k\}$ is related to the property of B to be a complex manifold.

To summarize the geometry of supersymmetric backgrounds with $SU(2)$-invariant spinors, we have found that the spacetime is a principal bundle $P(\mathcal{H}, B, \pi)$ equipped with a connection λ. The structure constants of the six-dimensional group $\mathcal{H}$ are constrained by (10.6). The curvature of λ is a self-dual two-form. The base space is a four-dimensional, conformally balanced HKT manifold. The metric and torsion are given in (10.14) and the dilaton Φ is a function of B.

10.1.4 Field equations and examples

The integrability conditions of the Killing spinor equations imply that all field equations are satisfied provided that $BH = BF = 0$ as in the previous cases with compact stability subgroups. In addition, one can show that $dH = \mathcal{F} \wedge \mathcal{F} + \pi^* d\tilde{H}$. This expression is similar to those in the G_2 and $SU(3)$ cases. If $\mathcal{H}$ is abelian, it is straightforward to introduce coordinates and write explicit expressions for the metric and torsion. Since we have done this for the $SU(3)$ case and the analysis is very similar, we shall not repeat the various formulae here. One can also easily construct the examples with non-vanishing null rotation which have also been considered in [47]. An example of a background with $SU(2)$-invariant spinors and eight supersymmetries is the NS5-brane [8].

10.2 Backgrounds with $\mathbb{R}^8$-invariant spinors

10.2.1 Supersymmetry conditions

As we have already explained, the Killing spinors can be chosen as

$$\begin{aligned} &\epsilon_1 = 1 + e_{1234}\ , \quad \epsilon_2 = i(1 - e_{1234})\ , \quad \epsilon_3 = e_{12} - e_{34}\ , \quad \epsilon_4 = i(e_{12} + e_{34})\ , \\ &\epsilon_5 = e_{13} + e_{24}\ , \quad \epsilon_6 = i(e_{13} - e_{24})\ , \quad \epsilon_7 = e_{23} - e_{14}\ , \quad \epsilon_8 = i(e_{23} + e_{14})\,. \end{aligned} \tag{10.15}$$

Observe that the above spinors can also be thought of as spanning the real chiral representation $\Delta^+_{\mathbf{8}}$ of $Spin(8)$. The gravitino Killing spinor equation implies that the connection $\hat{\nabla}$ takes values in $\mathbb{R}^8$, i.e. that

$$\hat{\Omega}_{A,ij} = \ , \quad \hat{\Omega}_{A,+B} = 0\ , \qquad i,j = 1,2,3,4\ ,6,7,8,9 \tag{10.16}$$

i.e. the only non-vanishing components of the connection are $\hat{\Omega}_{A,-i}$. The dilatino Killing spinor equation in addition implies that

$$
\begin{aligned}
&\partial_+\Phi = 0\,, \quad \partial_i\Phi - \frac{1}{2}H_{-+i} = 0\,,\\
&H_{ijk} = 0\,, \quad H_{+ij} = 0\,.
\end{aligned}
\tag{10.17}
$$

We shall next investigate the conditions on the geometry of spacetime.

10.2.2 Solution of the Killing spinor equations

It is clear from (10.16) that the connection $\hat{\Omega}$ takes values in $\mathbb{R}^8$ and so $\mathrm{hol}(\hat{\nabla}) \subseteq \mathbb{R}^8$. The forms of the Killing spinor bilinears can be written as

$$
\alpha = \mathrm{e}^- \wedge \phi\,, \qquad \phi \in \Lambda^{\mathrm{ev}+}(\mathbb{R}^8)\,, \tag{10.18}
$$

where $\Lambda^{\mathrm{ev}+}(\mathbb{R}^8) = \Lambda^0(\mathbb{R}^8) \oplus \Lambda^2(\mathbb{R}^8) \oplus \Lambda^{4+}(\mathbb{R}^8)$ and $\Lambda^{4+}(\mathbb{R}^8)$ denotes the subspace of self-dual four-forms and $\mathbb{R}^8 = \mathbb{R} < e^1, \dots, e^4, e^6, \dots, e^9 >$. As in the previous cases, the last condition in (10.16) implies that $\kappa = \mathrm{e}^-$ is a parallel null one-form associated with a Killing vector field X. In addition the last condition in (10.17) implies that X preserves the $\mathbb{R}^8$-structure, i.e.

$$
\mathcal{L}_X \alpha = 0\,, \tag{10.19}
$$

where α is any form Killing spinor bilinear.

The only non-vanishing components of H are H_{-ij} and H_{-+i}. Unlike previous $K \ltimes \mathbb{R}^8$ cases, H is determined in terms of the Levi-Civita connection and the dilaton. In particular, the conditions of the dilatino Killing spinor equation (10.17) express H_{-+i} in terms of Φ and the condition $\hat{\Omega}_{-,ij} = 0$ of the gravitino Killing spinor equation gives $H_{-ij} = 2\Omega_{-,ij}$. One can also adapt coordinates along the parallel vector field X, $X = \partial/\partial u$, and write the metric and torsion as

$$
\begin{aligned}
ds^2 &= 2(dv + m_I dy^I)(du + V dv + n_I dy^I) + \gamma_{IJ} dy^I dy^J\\
H &= \mathrm{e}^+ \wedge d\kappa + \Omega_{-,ij}\mathrm{e}^- \wedge e^i \wedge e^j\,.
\end{aligned}
\tag{10.20}
$$

Next consider the torsion free conditions for the frame $(\mathrm{e}^-, \mathrm{e}^+, e^i)$ which we can introduce as in (5.23). In particular, the torsion free conditions for e^- and e^i imply that there are functions $m = m(v, y)$ and $e^i = e^i(v, y)$ such that

$$
m_I = \partial_I m\,, \qquad e^i_J = \partial_J e^i(v, y)\,. \tag{10.21}
$$

In addition since $d\kappa = dm$, we have from the dilatino Killing spinor equation that

$$
\partial_I(2\Phi + \partial_v m) - 2m_I \partial_v \Phi = 0\,. \tag{10.22}
$$

If the rotation of X does not vanish, $d\kappa = d\mathrm{e}^- \neq 0$, it is not apparent that there is a diffeomorphism which preserves the form of the metric in (10.20) and transforms the "transverse" metric $\gamma_{IJ} dy^I dy^J$ to that of flat space. However if $d\kappa = d\mathrm{e}^- = 0$, the dilaton

$\Phi = \Phi(v)$, and one can perform the diffeomorphism $u = u, v = v$ and $y^i = e^i(v, y^I)$ which preserves the form of the metric in (10.20) and transforms the transverse part of the metric to that of flat space. In such a case, the solution of the Killing spinor equations can be written as

$$
\begin{aligned}
ds^2 &= 2dv(du + V dv + n_i dy^i) + \delta_{ij} dy^i dy^j \ , \\
H &= \Omega_{-,ij} dv \wedge dy^i \wedge dy^j \ , \qquad \Phi = \Phi(v) \, .
\end{aligned}
\tag{10.23}
$$

As in all previous $K \ltimes \mathbb{R}^8$ cases, the spacetime can be interpreted as a two parameter family of an eight-dimensional manifold. This can be done by introducing a frame (E^-, E^+, E^i) as in (5.31). If the rotation of X does not vanish, $d\kappa \neq 0$, then the deformed submanifold B although it admits a $\{1\}$-structure it is not compatible with the induced $\hat{\tilde{\nabla}}$ connection. However, if the rotation vanishes, then B is locally isomorphic to $\mathbb{R}^8$ as we have shown in (10.23).

10.2.3 Field equations and examples

One can show using the integrability conditions of the Killing spinor equations that the only field equations that need to be imposed in addition to the Bianchi identities $BH = 0, BF = 0$ are $E_{--} = 0$, $LH_{-A} = 0$ and $LF_- = 0$. This is similar to all the previous $K \ltimes \mathbb{R}^8$ backgrounds as might have been expected.

An example of a background with $\mathbb{R}^8$-invariant spinors and $N = 8$ supersymmetries is that of the fundamental string solution of [52]. The non-vanishing fields are

$$
\begin{aligned}
&ds^2 = 2h^{-1} dx dy + ds^2(\mathbb{R}^8) \ , \\
&H = -dx \wedge dy \wedge dh^{-1} \ , \\
&e^{2\Phi} = h^{-1} \ ,
\end{aligned}
\tag{10.24}
$$

where h is a harmonic function on $\mathbb{R}^8$. To see this, one performs the coordinate transformation

$$
u = x \ , \quad v = h^{-1} y \ , \tag{10.25}
$$

and the metric and torsion can be rewritten as

$$
\begin{aligned}
ds^2 &= 2(dv + v h^{-1} dh) du + ds^2(\mathbb{R}^8) \ , \\
H &= h^{-1} du \wedge dv \wedge dh = du \wedge d(v h^{-1} dh) \, .
\end{aligned}
\tag{10.26}
$$

Setting $\mathrm{e}^- = dv + v h^{-1} dh$ and $\mathrm{e}^+ = du$, the metric and torsion have the form of (10.20). In particular observe that (10.22) is satisfied.

11. Parallelizable string backgrounds

As we have mentioned in the introduction, if the Killing spinors have stability subgroup $\{1\}$, then the background is parallelizable with respect to the $\hat{\nabla}$ connection, i.e. $\hat{R} = 0$. In addition the gauge connection is flat, i.e. $F = 0$. In this section, we shall show that backgrounds for with $\hat{R}=0$ are group manifolds provided that H is closed at the 0-th order in α'. For this, we write the expression for the $\hat{R}$ curvature in terms of the Riemannian curvature R as

$$\hat{R}_{MN,RS} = R_{MN,RS} - \frac{1}{2}\nabla_M H_{NRS} + \frac{1}{2}\nabla_N H_{MRS} + \frac{1}{4} H_{RML} H^L{}_{NS} - \frac{1}{4} H_{RNL} H^L{}_{MS} \,, \tag{11.1}$$

where $M, N, \ldots = 0, \ldots, 9$ are coordinate indices. Skew-symmetrizing in all four indices and using $dH = 0$, we find that

$$H_{L[MN} H^L{}_{RS]} = 0 \tag{11.2}$$

which can be rewritten as

$$H_{LM[N} H^L{}_{RS]} = 0 \,. \tag{11.3}$$

Skew-symmetrizing (11.1) in the N, R, S indices, we get that

$$-\nabla_M H_{NRS} + \nabla_{[N} H_{RS]M} = 0 \tag{11.4}$$

which together with the closure of H gives

$$\nabla_M H_{NRS} = 0 \,. \tag{11.5}$$

Therefore the spacetime admits a parallel three-form which satisfies the Jacobi identity and so the spacetime is a ten-dimensional Lorentzian group manifold. Of course Lorentzian group manifolds admit sixteen parallel spinors with respect to the $\hat{\nabla}$ which can be identified with the left-invariant connection. In addition if we demand that the parallel spinors are also Killing, then the background is maximally supersymmetric and so the spacetime is locally isometric to Minkowski space [34].

The field equations impose additional conditions on backgrounds for which $\hat{R} = 0$. In particular, we find that the gravitino and two-form gauge potential field equations imply that

$$\nabla_M \partial_N \Phi = 0 \tag{11.6}$$

and

$$\partial_M \Phi g^{MN} H_{NRS} = 0 \,. \tag{11.7}$$

In the case that Φ is constant, the dilatino Killing spinor equation becomes

$$H_{MNR} \Gamma^{MNR} \epsilon = 0 \,. \tag{11.8}$$

This gives, after using the Jacobi equation, that

$$H_{MNR} H^{MNR} = 0 \,. \tag{11.9}$$

Therefore H is null. Since H is null, the spacetime admits at least eight Killing spinors.

Next let us turn to the case that Φ is not constant. In such case, (11.6) implies that $X^M = g^{MN}\partial_N\Phi$ is parallel. There are two cases to consider, either $X^2 = 0$, i.e. X is null, or $X^2 = \text{const}$ and so X is either timelike or spacelike. In both cases, using (11.7) and the Jacobi identity for H, we find that the dilatino Killing spinor equation implies that

$$X_M X^M - \frac{1}{24} H_{MNR} H^{MNR} = 0\,. \tag{11.10}$$

Therefore if X is null, then H is also null and the spacetime admits at least eight Killing spinors which satisfy $d\Phi\,\epsilon = 0$. If X is time-like, the condition (11.7), also written as $i_X H = 0$, implies that H is spacelike $H^2 > 0$ and so (11.10) cannot be satisfied. There are no parallelizable supersymmetric backgrounds, $\hat{R} = 0$, for which the dilaton can be identified with a time coordinate on the spacetime. The only remaining possibility is X space-like. Such supersymmetric backgrounds are known to exist like for example $\mathbb{R}^{5,1} \times U(1) \times SU(2)$. The dilaton is identified with the coordinate along the space-like $U(1)$ direction.

12. Common sector of type II supergravities

12.1 Supersymmetric backgrounds

The Killing spinor equations of the common sector of type II supergravities are

$$\begin{aligned} \nabla^{\pm}\epsilon_{\pm} &= 0\,, \\ (\Gamma^M\partial_M\Phi \mp \frac{1}{12}\Gamma^{MNP}H_{MNP})\epsilon_{\pm} &= 0\,, \end{aligned} \tag{12.1}$$

where $\nabla^{\pm} = \nabla \pm \frac{1}{2}H$ and $\epsilon_{\pm}$ are Majorana-Weyl spinors of the same (IIB) or opposite (IIA) chiralities. The field equations are the same as those of the metric and NS⊗NS two-form gauge potential of the heterotic string. In addition, $dH = 0$.

The Killing spinor equations of the common sector resemble those of the gravitino and dilatino of the heterotic supergravity. Observe for example that $\nabla^+ = \hat{\nabla}$. The common sector of type II supergravities is a consistent truncation of type II supergravities. Therefore it can be thought as a special case of eleven-dimensional and IIB supergravities. Because of this, we shall not investigate the Killing spinor equations in detail. Instead, we shall focus on some properties of the common sector supersymmetric backgrounds that follow from those of the supersymmetric backgrounds of the heterotic string that we have presented. In particular, we shall examine the common sector of IIB supergravity in which case $\epsilon_{\pm}$ are both positive chirality spinors. It is worth mentioning that the supersymmetric configurations of the common sector of IIA supergravity should be treated separately from those of IIB supergravity because there are several differences. To mention one, the spinors $1 + e_{1234}$ and $e_1 - e_{234}$ of IIA supergravity have stability subgroup $G_2 \ltimes \mathbb{R}^8$ in $Spin(9,1)$. As we have seen IIB supergravity does not admit spinors with stability subgroup $G_2 \ltimes \mathbb{R}^8$. Therefore, it is expected that some of the geometries that appear in the common sector IIA supergravity are different from those that appear in IIB.

The relevant spinor bundle of the IIB common sector is $S^+ \oplus S^+$ and the gravitino Killing spinor equation is a parallel transport equation for the connection $\nabla^+ \oplus \nabla^-$. Thus the gravitino Killing spinor equations are associated with a $Spin(9,1) \times Spin(9,1)$ connection. However, the gauge group that preserves the Killing spinor equations is the same as in the heterotic case, i.e. it is $Spin(9,1)$. This is the main difference between the Killing spinor equations of the common sector and those of the heterotic string. Because the gauge group is the proper diagonal subgroup of $Spin(9,1) \times Spin(9,1)$, it has many more orbits in the space of spinors than those of the heterotic string. As a result there are many more cases to consider.

The Killing spinors ϵ of the common sector can be written as $\eta_1 \oplus \eta_2$. To proceed, let G^+ and G^- be the stability subgroups of the parallel spinors $\eta \oplus 0$ and $0 \oplus \eta$, respectively, and G be the stability subgroup of all parallel spinors, $G \subseteq G^+ \cap G^-$. It is again the case that the holonomy of the $\nabla^\pm$ connections should be a subgroup of the stability subgroups $G^\pm$ of the parallel spinors, i.e. $\mathrm{hol}(\nabla^\pm) \subseteq G^\pm$. The general strategy to analyze the supersymmetric backgrounds of the common sector is to first choose the parallel spinors of the type $\eta \oplus 0$ as in the heterotic case and then use the residual gauge symmetry G^+ to simplify the Killing spinors of the type $0 \oplus \eta$, or vice versa. Without loss of generality, we may choose the Killing spinors $\epsilon \oplus 0$ as those of the heterotic supergravity. If one requires that there are parallel spinors with stability subgroup $G^+ = \{1\}$ or $G^- = \{1\}$, then either the curvature $R^+ = 0$ or $R^- = 0$. In such a case, the spacetime is a metric Lorentzian group. This follows from the results of the previous section. If both $G^+ = G^- = \{1\}$, then the Riemann curvature of the Levi-Civita connection vanishes, $R = 0$, and the spacetime is locally isometric to Minkowski space.

On the other hand, if either ∇^+ or ∇^- does not admit parallel spinors, then the analysis of the common sector reduces to that of the heterotic string for the connection with torsion that admits parallel spinors. In this case the heterotic supersymmetric backgrounds are "embedded" in the common sector.

It turns out that many common sector supersymmetric backgrounds admit a Killing spinor of the type spinor $\epsilon = \eta \oplus \eta$, i.e. η is parallel with respect to both ∇^+ and ∇^- connections. If this is the case, then η is also parallel with respect to the Levi-Civita connection. In particular, the Killing spinor equations imply that that

$$\begin{aligned} \nabla\eta = 0 \ , \quad H_{ABC}\Gamma^{BC}\eta &= 0 \ , \\ \Gamma^A\partial_A\Phi\eta &= 0 \, . \end{aligned} \tag{12.2}$$

The last condition follows from the dilatino Killing spinor equations. Since the stability subgroup of a single spinor in $Spin(9,1)$ is $Spin(7) \ltimes \mathbb{R}^8$, $G^+ = G^- = G = Spin(7) \ltimes \mathbb{R}^8$. So, $\mathrm{hol}(\nabla) \subseteq Spin(7) \ltimes \mathbb{R}^8$, i.e. the holonomy of the Levi-Civita connection is contained in $Spin(7) \ltimes \mathbb{R}^8$. Furthermore, the gauge symmetry of the Killing spinor equations can be used to set $\epsilon = 1 + e_{1234}$. As a result, one can use the results of $N = 1$ heterotic string backgrounds to show that

$$\Omega_{A,+B} = 0 \ , \quad \Omega_{A,\alpha}{}^{\alpha} = 0 \ , \quad \Omega_{A,\alpha\beta} = \frac{1}{2}\Omega_{A,\gamma\delta}\epsilon_{\alpha\beta}{}^{\gamma\delta} \ ,$$

$$
\begin{aligned}
&H_{A+B} = 0 \ , \quad H_{A\alpha}{}^{\alpha} = 0 \ , \quad H_{A\alpha\beta} = \frac{1}{2} H_{A\gamma\delta} \epsilon_{\alpha\beta}{}^{\gamma\delta} \ , \\
&\partial_+ \Phi = \partial_\alpha \Phi = 0 \ , \quad \alpha, \beta = 1, 2, 3, 4 \, .
\end{aligned} \tag{12.3}
$$

These imply that $\mathrm{hol}(\nabla) \subseteq Spin(7) \ltimes \mathbb{R}^8$, as we have already mentioned, $H_{-ij} \in \Lambda^2_{\mathbf{21}}$ and $H_{ijk} = 0$. The rotation of the associated null Killing vector field vanishes. As a result, the spacetime admits Penrose coordinates. One can also see that the Lorentzian deformation family is that of a $Spin(7)$ manifold, i.e. the Levi-Civita connection of B has holonomy contained in $Spin(7)$.

Next, we shall use the general results above to investigate common sector backgrounds with one and two supersymmetries. For $N = 1$, either ∇^+ or ∇^- admits a parallel spinor ϵ. This implies that either $\mathrm{hol}(\nabla^+) \subseteq Spin(7) \ltimes \mathbb{R}^8$ or $\mathrm{hol}(\nabla^-) \subseteq Spin(7) \ltimes \mathbb{R}^8$. Suppose that $\mathrm{hol}(\nabla^+)$ admits the parallel spinor ϵ. Since ϵ is Killing by assumption, it also solves the dilatino Killing spinor equation. The geometry of spacetime is that described in the case of $N = 1$ heterotic string backgrounds. The connection ∇^- may not admit parallel spinors, i.e. the holonomy of ∇^- is not contained in $Spin(7) \ltimes \mathbb{R}^8$. However, if it admits parallel spinors, they do not solve the dilatino Killing spinor equation.

To examine backgrounds with two supersymmetries, we again use the results we have derived in the context of heterotic string. There are several cases to consider. In the first case both Killing spinors are parallel with respect to either the ∇^+ or ∇^- connection. Without loss of generality, we can assume that both spinors are parallel with respect to the ∇^+ connection. There are two such cases to consider with stability subgroups $G = G^+ = SU(4) \ltimes \mathbb{R}^8$ and $G = G^+ = G_2$. The geometry of the spacetime is that of the $N = 2$ heterotic string backgrounds. Next suppose that one of the Killing spinors is parallel with respect to the ∇^+ connection and the other is parallel with respect to the ∇^- connection. In this case $G^+ = G^- = Spin(7) \ltimes \mathbb{R}^8$ and so the holonomy of both $\nabla^\pm$ connections is contained in $Spin(7) \ltimes \mathbb{R}^8$. The first Killing spinor can always to be chosen as $\epsilon_1 = f(1 + e_{1234})$. In such a case, one can show that the second Killing spinor can be chosen either as $\epsilon_2 = g_1(1 + e_{1234})$ or $\epsilon_2 = g_1(1 + e_{1234}) + i g_2(1 - e_{1234})$ or $\epsilon_2 = g(e_{15} + e_{2345})$. The argument is similar to the one we have used for the heterotic string backgrounds with two supersymmetries. The case for which both Killing spinors point to the same direction has already been investigated above and the supersymmetry conditions have been summarized in (12.3). The supersymmetry conditions for the remaining two cases can also be read from those of $N = 2$ heterotic string backgrounds. However for the second spinor, one has to alter appropriately the sign of the terms containing the flux H in the Killing spinor equations. The analysis is routine and we shall not present the results here. It is apparent though that one must consider many more cases of supersymmetric backgrounds in context of the common sector than those that appear in heterotic supergravity. In the table 12.1 we summarize the stability subgroups of the spinors of $N = 2$ IIB common sector backgrounds.

The parallel forms of string theory backgrounds are associated with conserved currents of the worldvolume action which described the propagation of (super)strings in such backgrounds [53]. Thus for any of the parallel forms we have presented in the supersymmetric heterotic and common sector backgrounds, there is an associated conserved current. In particular, we have shown that all supersymmetric heterotic and common sec-

$N=2$	G^+	G^-	G
	$SU(4)\ltimes\mathbb{R}^8$	-	$SU(4)\ltimes\mathbb{R}^8$
	G_2	-	G_2
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$SU(4)\ltimes\mathbb{R}^8$
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	G_2

Table 3: There are five classes of IIB common sector backgrounds with two supersymmetries. These are denoted with the stability subgroups G^+, G^- and G of the Killing spinors. In all cases $\mathrm{hol}(\nabla^\pm)\subseteq G^\pm$. The entries $-$ denote the cases for which the sector associated with the ∇^- connection does not admit Killing spinors.

tor backgrounds admit at least one parallel null vector field. Without loss of generality let $\nabla^+\kappa=0$, $\nabla^+=\hat\nabla$. Then the bosonic string with equations of motion $\nabla^+_+\partial_-Y^M=0$, where Y is the embedding map of the string worldvolume into spacetime and $\sigma^\pm$ are lightcone worldvolume coordinates, has a conserve current $\kappa_M\partial_-Y^M$, $\partial_+(\kappa_M\partial_-Y^M)=0$. Therefore $\kappa_M\partial_-Y^M=f(\sigma^-)$. It is known that the bosonic string action is invariant under the conformal transformations $\delta Y^M=a(\sigma^+)\partial_+Y^M+b(\sigma^-)\partial_-Y^M$, where a,b are the infinitesimal parameters. It is easy to see that choosing $f(\sigma^-)$ to be constant, one can gauge fix the conformal transformations associated with the parameter $b(\sigma^-)$. Similarly, if the one-form κ' is parallel with respect to the ∇^- connection, the current $\kappa'_M\partial_+Y^M$ is conserved $\partial_-(\kappa'_M\partial_+Y^M)=0$ and this can be used to gauge fix the conformal transformations associated with the infinitesimal parameter $a(\sigma^+)$.

13. Conclusions

We have specified the geometry of the supersymmetric heterotic string backgrounds (in the lowest order in α') for which all the parallel spinors of the connection $\hat\nabla$ with torsion given by the $NS\otimes NS$ three-form field strength are also Killing. We have also determined the field equations that are implied by the Killing spinor equations in all cases. We have found that there are two classes of backgrounds the null and timelike. The Killing spinors of null backgrounds are chiral which respect to a suitable $Spin(8)$ chirality projection or equivalently admit a single null $\hat\nabla$-parallel vector field. The stability subgroups of the Killing spinors in $Spin(9,1)$ are $K\ltimes\mathbb{R}^8$ for $K=Spin(7)$ $(N=1)$, $K=SU(4)$ $(N=2)$, $K=Sp(2)$ $(N=3)$, $K=SU(2)\times SU(2)$ $(N=4)$ and $K=\{1\}$ $(N=8)$, where N denotes the number of Killing spinors. We have shown that the spacetime is a suitable two-parameter Lorentzian family of an eight-dimensional manifold B with a K-structure. If the rotation of the null vector field vanishes, then B admits a metric connection, $\hat{\tilde\nabla}$ with skew-symmetric torsion on B compatible with an integrable conformally balanced K-structure on B, and so in particular $\mathrm{hol}(\hat{\tilde\nabla})\subseteq K$.

The Killing spinors of timelike backgrounds are not chiral which respect to a suitable $Spin(8)$ chirality projection or equivalently admit a time-like $\hat\nabla$-parallel vector field. The stability subgroups of the Killing spinors in $Spin(9,1)$ are G_2 $(N=2)$, $SU(3)$ $(N=4)$,

$SU(2)$ ($N = 8$) and $\{1\}$ (N=16). Assuming that the vector fields constructed from spinor bilinears close under Lie brackets, we have shown that the spacetime is locally a principal bundle, $P(\mathcal{H}, B, \pi)$, whose fibre directions are the orbits of the parallel vector fields and the base space is a manifold with a G_2 ($n = 7$) , $SU_c(3)$ ($n = 6$) and $SU(2)$ ($n = 4$)-structure, respectively, where $n = \mathrm{dim}B$. We have described the geometry of the spacetime of all supersymmetric backgrounds in terms of principal bundle data.

We also applied some of our results to the supersymmetric configurations of the common sector of type II supergravities. We have found that there are some differences between the properties of IIA and IIB supersymmetric common sector backgrounds. We also determined the conditions for the common sector IIB backgrounds with two supersymmetries. A consequence of our results is that *all* supersymmetric common sector and heterotic string backgrounds admit a null $\hat{\nabla}$-parallel vector field. This may be used to lightcone gauge fix the (super)conformal gauge symmetry of strings propagating in such backgrounds.

We have not investigated in detail the timelike backgrounds for which the set of the vector fields constructed from spinor bilinears does not close under Lie brackets. However, we have shown that the commutator vector field $[X, Y]$ of any two $\hat{\nabla}$-parallel vectors X, Y is also $\hat{\nabla}$-parallel. Therefore there are several possible geometric structures for the spacetime ranging from a principal bundle, which we have mentioned above, to a Lorentzian Lie group.

It is well-known the field equations of the heterotic string contain higher curvature correction terms. These modify the field equations of the supergravity theory that we have investigated. It has been shown in [10] that for certain supersymmetric backgrounds with $SU(3)$-invariant spinors, these higher order curvature correction terms are necessary for consistency with the heterotic anomaly cancelation mechanism. It would be of interest to find whether this persists to all supersymmetric backgrounds that we have analyzed.

Another class of supersymmetric heterotic backgrounds that we have not investigated are those for which the number of Killing spinors is less than the number of $\hat{\nabla}$-parallel spinors, i.e. some of the $\hat{\nabla}$-parallel spinors do not solve the dilatino Killing spinor equation. It is known such supersymmetric backgrounds exist. However, the analysis we have done for the gravitino Killing spinor equation in this paper still applies to this class of models. In particular, one can determine the stability subgroup in $Spin(9, 1)$ of the parallel spinors and construct the spacetime form spinor bilinears. Taking a basis in the space of parallel spinors $\{\eta_i\}$, one then can write the Killing spinors as $\epsilon_r = f_{ri}\eta_i$ and substitute them in the dilatino Killing spinors equation. Using the results we have collected in the appendices, one can derive linear systems similar to those which have been found in the context of IIB supergravity [31]. These linear systems can be solved to find the Killing spinors of such backgrounds.

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A. Spinors and forms

A.1 Spinors from forms

Spinors can be described in terms of forms. This construction is explained in, e.g. [54, 55] and it has been used in [29] in the context of manifolds with special holonomy. This description has been applied to the the Majorana-Weyl spinors of $Spin(9,1)$ in [38]. Here for completeness, we shall briefly summarize some of the aspects of this construction.

Consider the Euclidean space $U = \mathbb{R} < e_1, \dots, e_5 >$, where $e_1, \dots, e_5$ is an orthonormal basis. The space of Dirac spinors of $Spin(9,1)$ is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$. The gamma matrices are represented on Δ_c as

$$\begin{aligned} \Gamma_0\eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta \ , \qquad \Gamma_5\eta = e_5 \wedge \eta + e_5 \lrcorner \eta \ , \\ \Gamma_i\eta &= e_i \wedge \eta + e_i \lrcorner \eta \ , \qquad i = 1, \dots, 4 \\ \Gamma_{5+i}\eta &= i e_i \wedge \eta - i e_i \lrcorner \eta \, . \end{aligned} \tag{A.1}$$

Δ_c decomposes into two complex chiral representations according to the degree of the form $\Delta_c^+ = \Lambda^{\text{even}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\text{odd}}(U \otimes \mathbb{C})$. These are the complex Weyl representations of $Spin(9,1)$.

The gamma matrices $\{\Gamma_i; i = 1, \dots, 9\}$ are Hermitian and Γ_0 is anti-Hermitian with respect to the inner product

$$< z^a e_a, w^b e_b >= \sum_{a=1}^{5} (z^a)^* w^a \ , \tag{A.2}$$

on $U \otimes \mathbb{C}$ and then extended to Δ_c, where $(z^a)^*$ is the standard complex conjugate of z^a. The above gamma matrices satisfy the Clifford algebra relations $\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2\eta_{AB}$ with respect to the Lorentzian inner product as expected.

A $Spin(9,1)$ Majorana inner product is

$$B(\eta, \theta) =< B(\eta^*), \theta > \ , \tag{A.3}$$

where the map denoted with the same symbol $B = \Gamma_{06789}$. Observe that this Majorana inner product is skew-symmetric $B(\eta, \theta) = -B(\theta, \eta)$. B pairs the Δ_c^+ and Δ_c^- representations. Moreover, both Δ_c^+ and Δ_c^- are Lagrangian with respect to B, i.e. B restricted to either Δ_c^+ or Δ_c^- vanishes. The $Spin(9,1)$ Dirac inner product is

$$D(\eta, \theta) =< \Gamma_0\eta, \theta > \ . \tag{A.4}$$

It is well-known that $Spin(9,1)$ admits two inequivalent Majorana-Weyl representations. So it remains to impose the Majorana condition on the complex Weyl representations we have constructed above. The Majorana condition can be chosen as

$$\eta^* = -\Gamma_0 B(\eta) \ , \tag{A.5}$$

or equivalently

$$\eta^* = \Gamma_{6789}\eta\,. \tag{A.6}$$

Observe that this reality condition maps forms of even (odd)-degree to forms of even (odd)-degree and selects real subspaces Δ_{16}^+ and Δ_{16}^- in Δ_c^+ and Δ_c^-, respectively. These subspaces are the modules of the two inequivalent Majorana-Weyl representations of $Spin(9,1)$. For example 1 and e_{1234} are complex Weyl spinors while $1+e_{1234}$ and $i1-ie_{1234}$ are Majorana-Weyl, i.e. real chiral spinors.

The spacetime form bilinears associated with the spinors $\eta, \theta.$ are given as

$$\alpha(\eta,\theta) = \frac{1}{k!}B(\eta, \Gamma_{A_1\dots A_k}\theta)e^{A_1}\wedge\dots\wedge e^{A_k}\;, \qquad k=0,\dots,9\,. \tag{A.7}$$

If both spinors are of the same chirality, then it is sufficient to compute the forms up to degree $k\leq 5$. This is because the forms with degrees $k\geq 6$ are related to those with degrees $k\leq 5$ with a Hodge duality operation. The forms of middle dimension are either self-dual or anti-self-dual. If $\eta,\theta\in\Delta^+_{\mathbf{16}}$, then the non-vanishing forms are one-forms, three-forms and five-forms. In particular, one finds that $\alpha(\eta,\theta)=\alpha(\theta,\eta)$ for one- and five-forms, and $\alpha(\eta,\theta)=-\alpha(\theta,\eta)$ for three-forms.

In many computations that follow, it is convenient to use another basis in the space of spinors Δ_c. This is an oscillator basis given in terms of creation and annihilation operators. For this, first write

$$\Gamma_{\bar\alpha} = \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+5})\;, \qquad \Gamma_\pm = \frac{1}{\sqrt{2}}(\Gamma_5\pm\Gamma_0)\;, \qquad \Gamma_\alpha = \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+5})\,. \tag{A.8}$$

Observe that the Clifford algebra relations in the above basis are $\Gamma_A\Gamma_B+\Gamma_B\Gamma_A = 2g_{AB}$, where the non-vanishing components of the metric are $g_{\alpha\bar\beta}=\delta_{\alpha\bar\beta}, g_{+-}=1$. In addition we define $\Gamma^B = g^{BA}\Gamma_A$. The 1 spinor is a Clifford vacuum, $\Gamma_{\bar\alpha}1=\Gamma_+1=0$ and the representation Δ_c can be constructed by acting on 1 with the creation operators $\Gamma^{\bar\alpha},\Gamma^+$ or equivalently any spinor can be written as

$$\eta = \sum_{k=0}^{5}\frac{1}{k!}\,\phi_{\bar a_1\dots\bar a_k}\,\Gamma^{\bar a_1\dots\bar a_k}1\;, \qquad \bar a = \bar\alpha, +\,. \tag{A.9}$$

This is another manifestation of the relation between spinors and forms.

We conclude this section with our form conventions. A k-form α is denoted as $\alpha = \frac{1}{k!}\alpha_{i_1\dots i_k}e^{i_1}\wedge\dots\wedge e^{i_k}$ and so the components of the exterior derivative $d\alpha$ of α are $d\alpha_{i_1\dots i_k} = (k+1)\partial_{[i_1}\alpha_{i_2\dots i_{k+1}]}$. The inner product of two k-forms α and β is $(\alpha,\beta) = \frac{1}{k!}\alpha_{i_1\dots i_k}\beta_{j_1\dots j_k}g^{i_1j_1}\dots g^{i_kj_k}$, where g is the manifold metric. The Hodge dual $*\alpha$ of a k-form α is defined as $\alpha\wedge\beta = (*\alpha,\beta)\,d\mathrm{vol}$, where β is a $(n-k)$-form and $d$vol is the volume of the n-dimensional manifold. This gives that $*\alpha_{i_{k+1}\dots i_n} = \frac{1}{k!}\alpha_{j_1\dots j_k}\epsilon^{j_1\dots j_k}{}_{i_{k+1}\dots i_n}$. The inner derivation $i_I\alpha$ of a k-form α with an endomorphism I is $i_I\alpha = \frac{1}{(k-1)!}I^j{}_{i_1}\alpha_{ji_2\dots i_k}\,e^{i_1}\wedge\dots\wedge e^{i_k}$.

A.2 Spacetime forms from spinors

A.2.1 The $Spin(7)\ltimes\mathbb{R}^8$- and $SU(4)\ltimes\mathbb{R}^8$-invariant spinors

To compute the spacetime forms that are associated with the $Spin(7)\ltimes\mathbb{R}^8$- and $SU(4)\ltimes\mathbb{R}^8$-invariant spinors, it is sufficient to know the spacetime forms associated with the 1 and e_{1234}

spinors. This is because as we have seen 1 and e_{1234} span the $Spin(7)\ltimes\mathbb{R}^8$- and $SU(4)\ltimes\mathbb{R}^8$-invariant spinors. As a result, the spacetime forms associated with the $Spin(7)\ltimes\mathbb{R}^8$- and $SU(4)\ltimes\mathbb{R}^8$-invariant spinors are linear combinations of the forms associated with the 1 and e_{1234} spinors. Using (A.7), it is easy to find that the forms associated with the 1 and e_{1234} spinors are the following: A one-form

$$\kappa(e_{1234},1)=e^0-e^5\ , \tag{A.10}$$

a three-form

$$\xi(e_{1234},1)=-i(e^0-e^5)\wedge\omega\ , \tag{A.11}$$

and five-forms

$$\begin{aligned}&\tau(1,1)=(e^0-e^5)\wedge\chi\ ,\quad \tau(e_{1234},e_{1234})=(e^0-e^5)\wedge\chi^*\ ,\\&\tau(e_{1234},1)=-\frac{1}{2}(e^0-e^5)\wedge\omega\wedge\omega\ ,\end{aligned} \tag{A.12}$$

where

$$\begin{aligned}&\omega=-(e^1\wedge e^6+e^2\wedge e^7+e^3\wedge e^8+e^4\wedge e^9)\ ,\\&\chi=(e^1+ie^6)\wedge(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\,.\end{aligned} \tag{A.13}$$

A.2.2 The G_2-invariant spinors

The G_2 invariant spinors are linear combinations of 1, e_{1234}, e_{15} and e_{2345} spinors. The spacetime form bilinears associated with 1 and e_{1234} have been given in the previous section. Here we shall compute the spacetime forms associated with the rest of the spinor bilinears. In particular, we have the one-forms

$$\begin{aligned}&\kappa(1,e_{2345})=-e^1-ie^6\ ,\quad \kappa(e_{1234},e_{15})=-e^1+ie^6\ ,\\&\kappa(e_{2345},e_{15})=e^0+e^5\ ,\end{aligned} \tag{A.14}$$

the three-forms

$$\begin{aligned}\xi(1,e_{15})&=\hat\chi\ ,\\ \xi(1,e_{2345})&=-i(e^1+ie^6)\wedge\hat\omega+(e^1+ie^6)\wedge e^0\wedge e^5\ ,\\ \xi(e_{1234},e_{15})&=i(e^1-ie^6)\wedge\hat\omega+(e^1-ie^6)\wedge e^0\wedge e^5\ ,\\ \xi(e_{1234},e_{2345})&=\hat\chi^*\ ,\\ \xi(e_{2345},e_{15})&=-i(e^0+e^5)\wedge\hat\omega-i(e^0+e^5)\wedge e^1\wedge e^6\ ,\end{aligned} \tag{A.15}$$

where

$$\begin{aligned}\hat\omega&=-(e^2\wedge e^7+e^3\wedge e^8+e^4\wedge e^9)\ ,\\ \hat\chi&=(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\ ,\end{aligned} \tag{A.16}$$

and the five-forms

$$\tau(1,e_{15})=[-e^0\wedge e^5-ie^1\wedge e^6]\wedge\hat\chi\ ,$$

$$
\begin{aligned}
\tau(1, e_{2345}) &= (e^1 + ie^6) \wedge [\frac{1}{2}\hat{\omega} \wedge \hat{\omega} + i\hat{\omega} \wedge e^0 \wedge e^5] \ , \\
\tau(e_{1234}, e_{15}) &= (e^1 - ie^6) \wedge [\frac{1}{2}\hat{\omega} \wedge \hat{\omega} - i\hat{\omega} \wedge e^0 \wedge e^5] \ , \\
\tau(e_{1234}, e_{2345}) &= [-e^0 \wedge e^5 + ie^1 \wedge e^6] \wedge \hat{\chi}^* \ , \\
\tau(e_{2345}, e_{15}) &= (e^0 + e^5) \wedge [-\frac{1}{2}\hat{\omega} \wedge \hat{\omega} - \hat{\omega} \wedge e^1 \wedge e^6] \ , \\
\tau(e_{15}, e_{15}) &= -(e^0 + e^5) \wedge (e^1 - ie^6) \wedge \hat{\chi} \ , \\
\tau(e_{2345}, e_{2345}) &= -(e^0 + e^5) \wedge (e^1 + ie^6) \wedge \hat{\chi}^* . \qquad \text{(A.17)}
\end{aligned}
$$

A.2.3 The $N = 3$ case

The $Sp(2)$ invariant spinors are linear combinations of 1, e_{1234}, e_{12} and e_{34}. The spinor bilinears of 1 and e_{1234} have been computed already. Here, we shall give remaining forms. For this, let us set

$$
\begin{aligned}
\omega_1 &= -(e^1 \wedge e^6 + e^2 \wedge e^7), \qquad \omega_2 = -(e^3 \wedge e^8 + e^4 \wedge e^9) \ , \\
\chi_1 &= \left(e^1 + ie^6\right) \wedge \left(e^2 + ie^7\right), \qquad \chi_2 = \left(e^3 + ie^8\right) \wedge \left(e^4 + ie^9\right) . \qquad \text{(A.18)}
\end{aligned}
$$

Note that

$$
\omega = \omega_1 + \omega_2, \quad \chi = \chi_1 \wedge \chi_2 . \qquad \text{(A.19)}
$$

Then we find, the one-forms

$$
\kappa(e_{12}, e_{34}) = \kappa(e_{34}, e_{12}) = -(e^0 - e^5) \ , \qquad \text{(A.20)}
$$

the three-forms,

$$
\begin{aligned}
\xi(e_{12}, 1) &= -(e^0 - e^5) \wedge \chi_2 \ , \\
\xi(e_{34}, 1) &= -(e^0 - e^5) \wedge \chi_1 \ , \\
\xi(e_{1234}, e_{12}) &= -(e^0 - e^5) \wedge \chi_1^* \ , \\
\xi(e_{1234}, e_{34}) &= -(e^0 - e^5) \wedge \chi_2^* \ , \\
\xi(e_{34}, e_{12}) &= -i(e^0 - e^5) \wedge (\omega_1 - \omega_2) \ , \qquad \text{(A.21)}
\end{aligned}
$$

and five-forms

$$
\begin{aligned}
\tau(e_{12}, 1) &= i(e^0 - e^5) \wedge \omega_1 \wedge \chi_2 \ , \\
\tau(e_{34}, 1) &= i(e^0 - e^5) \wedge \omega_2 \wedge \chi_1 \ , \\
\tau(e_{12}, e_{1234}) &= i(e^0 - e^5) \wedge \omega_2 \wedge \chi_1^* \ , \\
\tau(e_{34}, e_{1234}) &= i(e^0 - e^5) \wedge \omega_1 \wedge \chi_2^* \ , \\
\tau(e_{12}, e_{12}) &= (e^0 - e^5) \wedge \chi_1^* \wedge \chi_2 \ , \\
\tau(e_{34}, e_{34}) &= (e^0 - e^5) \wedge \chi_1 \wedge \chi_2^* \ , \\
\tau(e_{12}, e_{34}) &= \frac{1}{2}(e^0 - e^5) \wedge (\omega_1 - \omega_2) \wedge (\omega_1 - \omega_2) . \qquad \text{(A.22)}
\end{aligned}
$$

A.2.4 $N = 4$, $SU(3)$

The relevant spinors are linear combinations of 1, e_{1234}, e_{15} and e_{2345}. The associated spacetime forms are given in section (A.2.2).

A.2.5 $N=4$, $(SU(2)\times SU(2))\ltimes\mathbb{R}^8$

We need to consider linear combinations of 1, e_{1234}, e_{12} and e_{34}. The spacetime forms are given in section (A.2.3).

A.2.6 $N=8$, $SU(2)$

The $SU(2)$-invariant Majorana-Weyl spinors are linear combinations of 1, e_{1234}, e_{12}, e_{34}, e_{15}, e_{25} e_{2345} and e_{1345}. There are five types of spinors that occur in $\Delta^+_{\mathbf{16}}$. These are

$$1\ ,\quad e_{1234}\ ,\quad e_{ij}\ ,\quad e_{i5}\ ,\quad e_{ijk5}\ ,\quad i,j,k=1,2,3,4\,. \tag{A.23}$$

We have already computed form bilinears of examples of all types. Because of this, one can compute the form spinor bilinears of the remaining pairs by appropriately relabelling the indices of the forms of the pairs we have already computed. We have done the computation but the result is not enlightening. Because of this we shall not explicitly list all the forms.

B. A linear system

In order to systematically solve the Killing spinor equations, we determine the action of the supercovariant derivative on the five types of spinors (A.23) and expand the results in the basis (A.9). This is similar to the calculation of IIB-supergravity and M-theory in [31]. We use the following conventions for the indices: $A,B,C\in\{+,-,\bar{1},..,\bar{4},1,..,4\}$, $\alpha,\beta\in\{1,..,4\}$ and $k,l,m,n\in\{1,..,4\}$ with the restriction $k,l,m,n\neq\alpha,\beta$. The Greek indices are not subject to the sum convention. In particular, we find

$$\frac{1}{4}\hat{\Omega}_{A,BC}\Gamma^{BC}1=\ \frac{1}{2}\left(\hat{\Omega}_{A,k}{}^{k}+\hat{\Omega}_{A,-+}\right)1+\frac{1}{4}\hat{\Omega}_{A,\bar{k}\bar{l}}\Gamma^{\bar{k}\bar{l}}1+\frac{1}{2}\hat{\Omega}_{A,+\bar{k}}\Gamma^{+\bar{k}}1\ , \tag{B.1}$$

$$\begin{aligned}\frac{1}{4}\hat{\Omega}_{A,BC}\Gamma^{BC}e_{1234}=&-\frac{1}{8}\hat{\Omega}_{A,kl}\epsilon^{kl}{}_{\bar{m}\bar{n}}\Gamma^{\bar{m}\bar{n}}1+\frac{1}{24}\hat{\Omega}_{A,+k}\epsilon^{k}{}_{\bar{l}\bar{m}\bar{n}}\Gamma^{+}\Gamma^{\bar{l}\bar{m}\bar{n}}1\\&+\frac{1}{2}\left(\hat{\Omega}_{A,-+}-\hat{\Omega}_{A,k}{}^{k}\right)e_{1234}\ ,\end{aligned} \tag{B.2}$$

$$\begin{aligned}\frac{1}{4}\hat{\Omega}_{A,BC}\Gamma^{BC}e_{\alpha\beta}=&-\hat{\Omega}_{A,\alpha\beta}1+\frac{1}{2}\hat{\Omega}_{A,\bar{k}\bar{l}}\epsilon^{\bar{k}\bar{l}\bar{\alpha}\bar{\beta}}e_{1234}\\&+\frac{1}{4}\left(\hat{\Omega}_{A,\bar{\alpha}\alpha}+\hat{\Omega}_{A,\bar{\beta}\beta}+\hat{\Omega}_{A,k}{}^{k}+\hat{\Omega}_{A,-+}\right)\Gamma^{\bar{\alpha}}\Gamma^{\bar{\beta}}1\\&+\frac{1}{2}\hat{\Omega}_{A,\bar{k}\alpha}\Gamma^{\bar{k}}\Gamma^{\bar{\beta}}1-\frac{1}{2}\hat{\Omega}_{A,\bar{k}\beta}\Gamma^{\bar{k}}\Gamma^{\bar{\alpha}}1\\&+\frac{1}{2}\hat{\Omega}_{A,+\alpha}\Gamma^{+}\Gamma^{\bar{\beta}}1-\frac{1}{2}\hat{\Omega}_{A,+\beta}\Gamma^{+}\Gamma^{\bar{\alpha}}1\\&+\frac{1}{4}\hat{\Omega}_{A,+\bar{k}}\Gamma^{+}\Gamma^{\bar{k}}\Gamma^{\bar{\alpha}\bar{\beta}}1\ ,\end{aligned} \tag{B.3}$$

$$\frac{1}{4}\hat{\Omega}_{A,BC}\Gamma^{BC}\Gamma^{+}\Gamma^{\bar{\alpha}}1=-\,2\hat{\Omega}_{A,-\alpha}1+\frac{1}{2}\left(\hat{\Omega}_{A,k}{}^{k}+\hat{\Omega}_{A,\bar{\alpha}\alpha}-\hat{\Omega}_{A,-+}\right)\Gamma^{+}\Gamma^{\bar{\alpha}}1$$

$$- \hat\Omega_{A,-\bar k}\Gamma^{\bar k}\Gamma^{\bar\alpha}1 - \hat\Omega_{A,\alpha\bar k}\Gamma^{+}\Gamma^{\bar k}1 + \frac{1}{4}\hat\Omega_{A,\bar k\bar l}\Gamma^{+}\Gamma^{\bar\alpha}\Gamma^{\bar k\bar l}1\ , \tag{B.4}$$

$$\begin{aligned}\frac{1}{4}\hat\Omega_{A,BC}\Gamma^{BC}\Gamma^{+}\Gamma^{\alpha}e_{1234} = &- 2\hat\Omega_{A,-\bar\alpha}e_{1234} - \frac{1}{4}\hat\Omega_{A,\bar\alpha k}\epsilon^{k}{}_{\bar\alpha\bar l\bar m}\Gamma^{+}\Gamma^{\bar\alpha\bar l\bar m}1\\ &+ \frac{1}{24}\left(-\hat\Omega_{A,k}{}^{k} + \hat\Omega_{A,\alpha\bar\alpha} - \hat\Omega_{A,-+}\right)\epsilon^{\alpha}{}_{\bar k\bar l\bar m}\Gamma^{+}\Gamma^{\bar k\bar l\bar m}1\\ &- \frac{1}{2}\hat\Omega_{A,kl}\epsilon^{\alpha kl}{}_{\bar m}\Gamma^{+}\Gamma^{\bar m}1 - \frac{1}{2}\hat\Omega_{A,-k}\epsilon^{\alpha k}{}_{\bar l\bar m}\Gamma^{\bar l\bar m}1\,.\end{aligned} \tag{B.5}$$

A similar analysis for the dilatino equation yields

$$\begin{aligned}\left(\partial_A\Phi\Gamma^A - \frac{1}{12}H_{ABC}\Gamma^{ABC}\right)1 = &\left(\partial_{\bar k}\Phi + \frac{1}{2}H_{\bar k}{}^{l}{}_{l} + \frac{1}{2}H_{+-\bar k}\right)\Gamma^{\bar k}1\\ &+ \left(\partial_{+}\Phi - \frac{1}{2}H_{+k}{}^{k}\right)\Gamma^{+}1 - \frac{1}{4}H_{+\bar k\bar l}\Gamma^{+}\Gamma^{\bar k\bar l}1\\ &- \frac{1}{12}H_{\bar k\bar l\bar m}\Gamma^{\bar k\bar l\bar m}1\ ,\end{aligned} \tag{B.6}$$

$$\begin{aligned}\left(\partial_A\Phi\Gamma^A - \frac{1}{12}H_{ABC}\Gamma^{ABC}\right)e_{1234} = &\ \frac{1}{6}H_{klm}\epsilon^{klm}{}_{\bar n}\Gamma^{\bar n}1\\ &+ \left(\partial_{+}\Phi + \frac{1}{2}H_{+k}{}^{k}\right)\Gamma^{+}e_{1234}\\ &+ \frac{1}{12}\left(\partial_k\Phi + \frac{1}{2}H_{kl}{}^{l} + \frac{1}{2}H_{+-k}\right)\epsilon^{k}{}_{\bar l\bar m\bar n}\Gamma^{\bar l\bar m\bar n}1\\ &+ \frac{1}{8}H_{+kl}\epsilon^{kl}{}_{\bar m\bar n}\Gamma^{+}\Gamma^{\bar m\bar n}1\ ,\end{aligned} \tag{B.7}$$

$$\begin{aligned}\left(\partial_A\Phi\Gamma^A - \frac{1}{12}H_{ABC}\Gamma^{ABC}\right)e_{\alpha\beta} = &-\frac{1}{2}H_{+\bar k\bar l}\epsilon^{\bar k\bar l\bar\alpha\bar\beta}\Gamma^{+}e_{1234} + H_{+\alpha\beta}\Gamma^{+}1 + H_{\bar k\alpha\beta}\Gamma^{\bar k}1\\ &+ \left(-\partial_\beta\Phi + \frac{1}{2}H_{\bar\alpha\alpha\beta} + \frac{1}{2}H_{\beta l}{}^{l} - \frac{1}{2}H_{+-\beta}\right)\Gamma^{\bar\alpha}1\\ &+ \left(\partial_\alpha\Phi - \frac{1}{2}H_{\bar\beta\beta\alpha} - \frac{1}{2}H_{\alpha l}{}^{l} + \frac{1}{2}H_{+-\alpha}\right)\Gamma^{\bar\beta}1\\ &+ \frac{1}{2}\left(\partial_{+}\Phi - \frac{1}{2}H_{+k}{}^{k} - \frac{1}{2}H_{+\bar\alpha\alpha} - \frac{1}{2}H_{+\bar\beta\beta}\right)\Gamma^{+}\Gamma^{\bar\alpha}\Gamma^{\bar\beta}1\\ &+ \frac{1}{2}\left(\partial_{\bar k}\Phi + \frac{1}{2}H_{\alpha\bar\alpha\bar k} + \frac{1}{2}H_{\beta\bar\beta\bar k} - \frac{1}{2}H_{\bar k l}{}^{l} + \frac{1}{2}H_{+-\bar k}\right)\Gamma^{\bar k}\Gamma^{\bar\alpha}\Gamma^{\bar\beta}1\\ &- \frac{1}{4}H_{\alpha\bar k\bar l}\Gamma^{\bar k\bar l}\Gamma^{\bar\beta}1 + \frac{1}{4}H_{\beta\bar k\bar l}\Gamma^{\bar k\bar l}\Gamma^{\bar\alpha}1\\ &+ \frac{1}{2}H_{+\alpha\bar k}\Gamma^{+}\Gamma^{\bar k}\Gamma^{\bar\beta}1 - \frac{1}{2}H_{+\beta\bar k}\Gamma^{+}\Gamma^{\bar k}\Gamma^{\bar\alpha}1\ ,\end{aligned} \tag{B.8}$$

$$\begin{aligned}\left(\partial_A\Phi\Gamma^A - \frac{1}{12}H_{ABC}\Gamma^{ABC}\right)\Gamma^{+}\Gamma^{\bar\alpha}1 = &\ \frac{1}{3}H_{\bar k\bar l\bar m}\epsilon^{\bar k\bar l\bar m\bar\alpha}\Gamma^{+}e_{1234} + 2H_{-\alpha\bar k}\Gamma^{\bar k}1\\ &+ \left(2\partial_{-}\Phi - H_{-\bar\alpha\alpha} - H_{-k}{}^{k}\right)\Gamma^{\bar\alpha}1\\ &+ \left(-2\partial_\alpha\Phi + H_{\alpha k}{}^{k} + H_{+-\alpha}\right)\Gamma^{+}1\\ &+ \left(\partial_{\bar k}\Phi + \frac{1}{2}H_{\alpha\bar\alpha\bar k} - \frac{1}{2}H_{+-\bar k} + \frac{1}{2}H_{l\bar k}{}^{l}\right)\Gamma^{+}\Gamma^{\bar\alpha}\Gamma^{\bar k}1\\ &+ \frac{1}{2}H_{\alpha\bar k\bar l}\Gamma^{+}\Gamma^{\bar k\bar l}1 - \frac{1}{2}H_{-\bar k\bar l}\Gamma^{\bar\alpha}\Gamma^{\bar k\bar l}1\ ,\end{aligned} \tag{B.9}$$

$$\begin{aligned}\left(\partial_A\Phi\Gamma^A - \frac{1}{12}H_{ABC}\Gamma^{ABC}\right)\Gamma^{+}\Gamma^{\alpha}e_{1234} = &\ \frac{1}{6}\left(\partial_{-}\Phi + \frac{1}{2}H_{-\bar\alpha\alpha} + \frac{1}{2}H_{-k}{}^{k}\right)\epsilon^{\alpha}{}_{\bar k\bar l\bar m}\Gamma^{\bar k\bar l\bar m}1\\ &- 2\left(\partial_{\bar\alpha}\Phi - \frac{1}{2}H_{+-\bar\alpha} + \frac{1}{2}H_{\bar\alpha k}{}^{k}\right)\Gamma^{+}e_{1234}\\ &- \frac{1}{2}\left(\partial_k\Phi + \frac{1}{2}H_{\bar\alpha\alpha k} - \frac{1}{2}H_{+-k} + \frac{1}{2}H_{kl}{}^{l}\right)\epsilon^{\alpha k}{}_{\bar l\bar m}\Gamma^{+}\Gamma^{\bar l\bar m}1\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2} H_{\bar{\alpha}kl} \epsilon^{kl}{}_{\bar{\alpha}\bar{m}} \Gamma^{+} \Gamma^{\bar{\alpha}} \Gamma^{\bar{m}} 1 + H_{-kl} \epsilon^{\alpha kl}{}_{\bar{m}} \Gamma^{\bar{m}} 1 \\
&+ \frac{1}{2} H_{-\bar{\alpha}k} \epsilon^{k}{}_{\bar{\alpha}\bar{l}\bar{m}} \Gamma^{\bar{\alpha}} \Gamma^{\bar{l}\bar{m}} 1 + \frac{1}{3} H_{klm} \epsilon^{klm\alpha} \Gamma^{+} 1 \,. \qquad \text{(B.10)}
\end{aligned}
$$

The above expressions can be used to construct the linear systems for those backgrounds for which not all parallel spinors are Killing. We have also used the above equations to determine the conditions on the geometry of spacetime for the supersymmetric backgrounds we have examined.

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